

ALTERNATIVE TESTS OF
K-FACTOR ARBITRAGE PRICING THEORY

BY

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The simple economic intuition behind the Arbitrage Pricing Theory (APT) has received considerable attention from theorists and empiricists alike as it provides a promising alternative to the mean-variance Capital Asset Pricing Model (CAPM). However, tests of the APT can be more fully implemented. This research provides an extended set of tests of the APT in a multi-period context.

In this regard, the convergence test of the Ross strict factor structure (SFS) and the Chamberlain and Rothschild (CR) approximate factor structure (AFS) as well as the multivariate linear regression test with the errors-in-variable approach are developed to test the APT with emphasis on the convergence and stationarity issue.

Specifically, a new approach is employed to examine whether the factor pattern matrices of Ross's SFS and CR's AFS converge as the number of assets grows. Employing the orthogonal procrustes rotation methodology (OPRM) and the well-known varimax rotation methodology (VRM), we found supporting evidence that two factor structures converge as the number of assets becomes large, implying that the variance of idiosyncratic risk is bounded.

In addition, a multi-period test of the k-factor APT is proposed dealing with the stability of the number of relevant economic or priced factors. The empirical results reveal that the stability of the k-factor structure is violated during the sample periods, from January 1960 to December 1983, indicating that the parsimonious aspects of the APT are weakened.

One important result of the APT is that there should be no other economic/financial variables which exert any additional influence on the expected returns once the effects of the factor loadings and/or the eigenvectors have been accounted for. This paper provides empirical evidence on this issue. It turns out that the extraneous variables including skewness and firm size do not exert any additional explanatory power after the adjustment of systematic risks. Also, the related January effect does not exhibit any particular systematic pattern which can distinguish it from the remaining eleven months when employing the multi-factor APT. However, when its own standard deviation is added to the APT equation, it seems to be a more important determinant of stock returns than factor risk premia.

CHAPTER I
INTRODUCTION AND BACKGROUND

Introduction

The two leading models for analyzing equilibrium asset pricing theory are the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT). The CAPM is based on mean-variance analysis which is introduced by Markowitz (1952, 1959) and Tobin (1958). Its development as a description of equilibrium in capital markets is due to Sharpe (1964), Lintner (1965), Mossin (1966) and Black (1972). It shows the relationship between mean-variance efficiency and the location of the market portfolio. In equilibrium, the expected return for security i is given by

$$E(R_i) = E(R_z) + [E(R_m) - E(R_z)] B_i \quad (1-1)$$

where $E(R_i)$ = the expected return on security i

$E(R_z)$ = the expected return on the zero-beta portfolio or
the risk-free rate if it exists

$E(R_m)$ = the expected return on the market portfolio
= $\text{cov}(R_i, R_m) / \text{var}(R_m)$

B_i = the beta coefficient for security i

Equation (1-1) states that the market portfolio must be efficient

in equilibrium if the assumption of homogeneous expectations holds. However, one of the valuable properties of the CAPM, its testability, is seriously questioned by Roll (1977) simply because the market portfolio is unobservable. Also, Roll shows that mean-variance efficiency of the market portfolio is mathematically equivalent to the linear relationship of expected return and beta. Thus, the empirical evidence on the market inefficiency could be simply due to the misspecification of the CAPM.

In this regard, recent research on asset pricing has tended to head toward multi-factor pricing models. Particularly, the Arbitrage Pricing Theory (APT) of Ross (1976, 1977) has gained wide attention from theorists and empiricists alike. Its simple theoretical developments and testable implications make it a reasonable alternative to the mean-variance CAPM. To introduce the Ross APT briefly, it may be helpful to illustrate the basic assumptions of the APT which are simpler than those of the CAPM:

- (i) There are many assets
- (ii) All individuals believe that security returns are generated by the same small number (k) of common factors and
- (iii) The market permits no unexploited arbitrage opportunities in equilibrium.

Based upon these simple assumptions, the APT indicates that the assets' expected returns are linear in the factor loadings.

$$E(R_i) = RP_0 + \sum_j RP_j b_{ij} \quad (1-2)$$

where RP_0 = the return on a zero-beta asset or the riskless

rate of return if there exists a risk-free asset

RP_j = a vector of risk premia

b_{ij} = a measure of the sensitivity of asset i to
the variations in factor j

This APT pricing equation (1-2) is motivated by the simple economic intuition that idiosyncratic noise terms represent diversifiable risk and therefore should not affect asset pricing. That intuition is similar to that of the CAPM. However, the APT does not require the identification and measurement of a market portfolio. In addition, it is often asserted that, not only are the assumptions of the APT less restrictive than those of the CAPM, but also it is testable on any subset of assets. For these reasons, texts of the APT have been examined by numerous authors to determine if the APT is a reasonable alternative or a superior model relative to the CAPM.

Motivation and Objectives

Ever since Ross (1976, 1977) introduced the APT, much attention has been devoted to tests of the APT. The proliferation of this literature can be traced to the well-publicized deficiencies in the popular Capital Asset Pricing Model (CAPM) and to the theoretical simplicity and empirical tractability of the APT.

Specifically, the study by Roll and Ross (RR) (1980) exemplified this literature by providing preliminary evidence consistent with the theory. However, more recently, a few serious studies challenged the views of Ross and RR. In

particular, Shanken (1982) challenged the views of Ross and RR that the APT is inherently more susceptible to empirical verification than the CAPM by arguing that any finite sum of squared deviations is clearly finite, implying that the boundary condition of the APT is not empirically testable.

Other serious studies include that of Chamberlain and Rothschild (CR) (1983) and that of Dhrymes, Friend, Gultekin, and Gultekin (DFGG) (1985b). Chamberlain et al rigorously show that the necessary and sufficient condition for returns to be linearly related to k factors is that k eigenvalues of the covariance matrix of returns grow large as the number of securities increases.

To examine the behavior of k exploding eigenvalues, CR argue that only principal component analysis (PCA) is needed in empirical work, implying that the maximum likelihood factor analysis (MLFA) employed by RR is not an inevitable methodology to test the APT. They further claim the fundamental assumption of Ross that the small number of economically relevant factors (k) represents the economy as overly strong. Instead, CR argue that an approximate K -factor structure is still sufficient for the Ross APT to hold.

Focusing on the different side of the APT, DFGG (1985b) questioned the appropriateness of the APT in a multi-period context. In particular, they dispute the stability of the number of priced factors over time by arguing that the ability of risk measures from one period differs from that of another period. Based upon this brief review of critical studies, it can be concluded that the APT has not been fully explored.

This study is motivated by the fact that the several aspects of the APT should be scrutinized more thoroughly. In particular, it seems that there exists a couple of urgent needs. First, one must find out a certain way of examining the boundary condition of the APT. In relation with Shanken's argument that the boundary condition of the APT is not empirically testable, Dybvig (1983) and Grinblatt and Titman (GT) (1983) independently derive the similar explicit bound of the APT prices.

However, even though the explicit bound of Dybvig and GT is conceptually sound, it is difficult to test the APT bound empirically because it is associated with the measure of relative risk aversion and the utility function of the representative agent. Therefore it seems almost compulsory to find an alternative way to test whether the variance of idiosyncratic risks is bounded as the number of assets grows. Second, as DFGG (1985b) correctly point out, since the previous tests of the APT have not been directly concerned with the dynamic properties of the model, it seems necessary to examine the multi-period validity of the APT more thoroughly. For example, if the number of factors responsible for the causative, concrete and economic components is large or changing frequently over time, the parsimonious aspects of the APT are weakened. The APT is a useful theory only if the underlying distribution of asset returns allows one to produce diversifiable risk with a representation having $k \ll n$ across assets and over time.

The main objective of this dissertation is two-fold. One is to derive the conditions which verify the boundary issue of the

residual variance condition of the APT and examine such conditions empirically. This important issue of testing the boundary condition of diversifiable risks of the APT is scrutinized by examining the convergence of the Ross strict factor structure (SFS) and the CR approximate factor structure (AFS). Another purpose of this paper is to test the multi-period behavior of the APT by developing multivariate linear regression with the errors-in-variable approach. It is shown that at any point in time, portfolio returns can be expressed as a linear projection on the historical expected returns and the expected returns of a reference portfolio at the same point in time. One other small objective of this paper is to examine the impacts of extraneous variables upon the APT pricing.

Contributions

The Ross APT is based on the economic intuition that the systematic risk is the prime driver for pricing assets in the economy and firm specific risk represents a diversifiable risk which should have a zero price in a market with no arbitrage opportunities.

To be specific, Ross (1976) shows that, in a market with no arbitrage possibility, there must exist $RP_0, RP_1, \dots, RP_k$ such that

$$\lim \sum_i (E_i - RP_0 - (\sum_j RP_j b_{ij}))^2 \quad (1-3)$$

Equation (1-3) is less than some finite value h as the number of assets increases without bound. However, even though most assets

can be priced so that

$$|E_i - RP_0 - (\sum_j RP_j b_{ij})| < A \text{ for } A > 0,$$

a finite number of assets can be priced such that

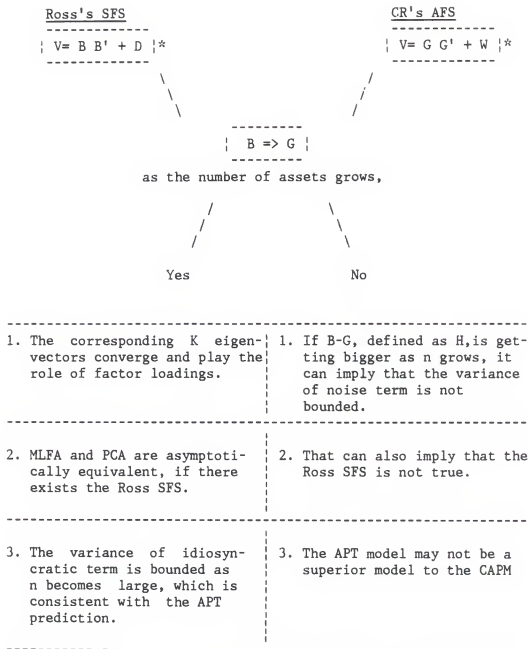
$$|E_i - RP_0 - (\sum_j RP_j b_{ij})| > A,$$

because the APT holds as an approximation. Of course, without a priori information, or further restrictive assumptions, one can not determine which assets and/or how many assets can fall in this class.

Hence, the examination of the convergence of the Ross SFS and the CR AFS is an alternative contribution of testing the economic intuition of the APT that the firm specific risk should not be priced in a market with no arbitrage opportunities, or alternatively, the variance of idiosyncratic risk should be bounded. Simplified intuition of the convergence tests of the APT is illustrated in Figure 1-1.

This issue of convergence tests of the Ross SFS and the CR AFS is tackled by employing the orthogonal procrustes rotation methodology (OPRM) and comparing its results with those of well-known varimax rotation methodology (VRM).

Another valuable property of the APT is that it can be easily extended to a multi-period framework (Ross 1976). However, the evidence on the stability of k-factor structure for a multi-period setting is at best mixed. For instance, DFGG (1985b) provide evidence that the number of priced factors found in the first period is significantly different from that of the subsequent period. Moreover, DFG (1984) and DFGG (1985a, 1985b)



Note: * $V = (n \times n)$ covariance matrix of $E[(R - E)(R - E)']$
 $B = (n \times k)$ matrix of factor loadings
 $D = (n \times n)$ diagonal matrix of asset variances
 $G = (n \times k)$ matrix of eigenvectors
 $W = (n \times n)$ positive semi-definite matrices with uniformly bounded eigenvalues

Figure 1-1
Simplified Intuition of the Convergence Tests of the APT

claim that since MLFA employed by RR and themselves is flawed, they can not definitely reject the APT model, even though their evidence is not favorable.

This paper provides some contribution to this literature by developing multivariate linear regression with the errors-in-variable approach to the multi-period context and employing PCA as well as MLFA to test the stability of the K-factor APT over time.

Overview

What follows is an outline of this dissertation. The literature of the APT is briefly reviewed in Chapter II. This chapter includes the review of the theoretical debates on the testability of the APT, the review of the empirical studies on the tests of the APT, and the review of the literature on the number of factors in the APT model.

Chapter III introduces the Ross strict factor structure (SFS), the CR approximate factor structure (AFS) and compares the two. This chapter provides the additional characteristics of the CR AFS and PCA which CR claim is enough alone for testing the APT. The subsequent section of the Chapter III examines the convergence between SFS and AFS in the multi-period setting by providing both the necessary and sufficient condition for obtaining the minimum difference between the factor pattern matrices of SFS and AFS. This task is accomplished by employing the orthogonal procrustes rotation methodology (OPRM) which is introduced by Green (1952) and comparing the OPRM with the

varimax rotation methodology (VRM) introduced by Kaiser (1958) and adopted in the APT testing literature by RR (1980). Chapter III also illustrates that the average statistics obtained from these OPRM and VRM should be getting smaller as the number of assets grows. Empirical results reveal that the average squared differences of the factor pattern matrices of MLFA and PCA are getting smaller when the number of assets grows. The results are consistent with the prediction of the APT in that the variance of idiosyncratic risks should be bounded.

Chapter IV presents the generalized Jobson's multivariate linear regression approach, and improved version of testing the APT using time-series data, emphasizing the stability of k-factor structure and that of a zero-beta rate over time. This issue is pursued by introducing the reference portfolio concept which is originally adopted by RR (1980). Specifically, it is shown that at any point in time, portfolio returns can be expressed as a linear projection on the historical expected returns and the expected returns of a reference portfolio at the same point of time. This is shown to be equivalent to testing the zero intercept and the coefficients that sum to one in the employed multivariate regression.

Furthermore, some implication of the linear combination of the APT is presented. Specifically, under certain conditions, any minimum variance portfolio can be expressed as a linear combination of the arbitrage portfolio and the minimum variance zero-beta portfolio.

The empirical results with regard to the stability tests of k-factor APT over time reveal that the stability of k-factor

structure is violated during the sample periods, from January 1960, to December 1983, indicating that the parsimonious aspects of the APT are weakened. In addition, the empirical results of the tests for the stationarity of a zero-beta rate or a risk-free rate show that the risk-free rate is stable only for a time period of 6 years, but not for a longer period of 12 years.

Chapter V presented a variety of evidence with regard to the tests on the economic content of the APT. This chapter examined whether the other financial variables including the own standard deviation, the skewness, and the firm size had additional explanatory power after the adjustment of systematic risks. It turns out that the extraneous variables, such as, the skewness and the firm size do not exert any additional explanatory power after the adjustment of k eigenvectors. Also, the related January effect does not exhibit any particular systematic pattern which can distinguish it from the remaining eleven months. However, when its own standard deviation is added to the APT equation, it seems to be a more important determinant of stock returns than factor risk premia.

Finally, the conclusions are contained in Chapter VI.

CHAPTER II LITERATURE REVIEW

Introduction

The Arbitrage Pricing Theory (APT) of Ross (1976, 1977) has received considerable attention from theorists and empiricists alike, since it provides a promising testable alternative to the mean-variance Capital Asset Pricing Model (CAPM). It has been often asserted that the APT not only offers a theoretically more robust and more testable alternative than the CAPM, but also holds in both single period and multi-period context.

Based upon this intuitive generality of the APT, a number of financial researchers have attempted to test the APT. To facilitate a logical review, the literature review is divided into three parts. In the first part, theoretical studies on the testability of the APT are reviewed. In particular, Shanken's (1982) main criticism of previous APT testing literature is presented. Shanken argues that the APT is in principle untestable because economically irrelevant repackaging of assets will cause the econometrician to identify a different factor structure with a lower dimension than what the same econometrician would have identified using data from the original assets. The interpretation of Shanken is shown to be incorrect

by Dybvig and Ross (DR) (1985) and Grinblatt and Titman (GT) (1985) among others.

The second part of this chapter reviews the current research on the empirical tests of the APT. While a series of theoretical papers investigate the validity of the APT and show that the APT works reasonably well in the large asset market at least in an approximation sense, a number of empirical papers examine the empirical relevance of APT and provide some mixed evidence. To be specific, Gehr (1978), RR (1980), Chen (1983), and Brown and Weinstein (1983) found the evidence to support the APT. However, Reinganum (1981), Dhrymes, Friend, and Gultekin (DFG) (1984), and Dhrymes, Friend, Gultekin and Gultekin (DFGG) (1985b) found the evidence inconsistent with the APT.

The third part of this chapter reviews the existing studies on the number of economic factors. To date, the voluminous studies of the APT have addressed the issue of the number of factors, because a fundamental assumption in the APT is that all security returns are generated by a small number (K) of factors. Among others, the studies of Kryzanowski and To (1983), DFG (1984), RR (1984), DFGG (1985a, 1985b) and Trzcinka (1986) are briefly presented. This issue is particularly important because if it is necessary to use a large number of factors or changing number of factors over time to produce zero idiosyncratic risk, then the APT is not an extremely useful theory in that the parsimonious aspect of the APT is lost.

The last part of this chapter summarizes the results of major studies.

Review of the Theoretical Studies on the Testability of the APT

Shanken (1982) challenges the view of Ross (1976) that the APT is inherently more testable than the CAPM by questioning a fundamental question of what it means to test the APT. Shanken criticizes a factor analysis as an inappropriate tool for testing the APT. He argues that factor analysis is merely concerned with statistical correlations and so lacks aggregate economic considerations. Using two repackaged securities, Shanken argues that the Ross APT does not imply an empirically testable bound on the sum of squared deviations and concludes that empirical tests of the APT should take into account the assumptions of theory and the resultant approximation error.

Regarding Shanken's criticism of factor analysis as an inappropriate tool for testing the APT, Grinblatt and Titman (GT) (1985b) argue that since the variance of Shanken's repackaged securities is of the order of the eigenvalues of the original covariance matrix, it directly follows that the k largest eigenvalues are infinite. Grinblatt et al further claim that since common factors are selected even in repackaged securities if samples are large enough by Maximum Likelihood Factor Analysis (MLFA) and/or Principal Component Analysis (PCA), the estimated factor loadings and/or eigenvectors will be consistent estimates of the factor sensitivities.

Dybvig and Ross (DR) (1985) emphasize the reasonableness of economic assumptions used in the derivation of theory for the fundamental assets in our economy. As the assumptions made in the APT model are not good assumptions for arbitrarily transformed assets, DR conclude that Shanken makes an error in

applying the APT to arbitrary portfolios. In this regard, Shanken's argument based on arbitrary portfolios loses its strength.

Another point emphasized by Shanken to question the testability of the APT is the resultant approximation error. Grinblatt and Titman (GT) (1983) and Dybvig (1983) derive an explicit bound on the deviation from the APT prices across assets and assert that the APT pricing relationship is a good approximation for the expected return of each traded security or portfolio in our economy.

Dybvig and Ross (1985) further dispute Shanken by arguing that the question of testability of a theory is independent of the question of the accuracy of the approximations used in its derivation. Therefore, Shanken's point of approximation error is irrelevant to the question of testability of the APT, and at worst case if it is relevant, the resultant approximation error is so small that the APT pricing is a good approximation in our economy. To put this matter different way, the APT is based on the view that many assets are close substitutes such that the idiosyncratic error terms represent diversifiable risk and so should not affect pricing.

On the other hand, Chamberlain and Rothschild (CR) (1983) prove that, for the APT to be testable on a subset, it is sufficient for the square of the factor pattern matrix, BB' , to have unbounded k eigenvalues with the remaining $(n-K)$ bounded eigenvalues. They also imply that although Shanken does not

explicitly restrict his analysis to small subsets of the economy, Shanken's degeneracy case cannot occur in large samples.

Similarly, Ingersoll (1984) shows the unbounded eigenvalues' characterization of significant factors so that there should be a k -factor structure where K is the number of unbounded eigenvalues of BB' of the subset. In sum, CR, DR, GT, and Ingersoll seem to agree that the APT is always testable on subsets.

Review of the Empirical studies on the Tests of the APT

This section presents the empirical tests of the APT by classifying these tests into five approaches, namely, the portfolio approach, security approach, multivariate linear regression approach, bilinear paradigm approach, and large asset approach.

First approach employed in the empirical tests of the APT is the portfolio approach used by Gehr (1978), RR (1980) and Reinganum (1981) among others. This approach can overcome the problem of inefficient estimation by grouping assets into different portfolios. However this approach has the problems associated with the determination of the number of factors with the comparison of factors among different groups.

Gehr (1978) is the first empirical test of the APT. By factor analyzing two distinct data sets including 24 industry index excess returns and 41 stock excess returns, he found that no more than three factors explained approximately 41 percent of the variance in returns, but only one of the factors was significant in the second step regression. However his conclusion is not generalizable to the universe of securities

because increasing the number of securities in sample may lead to a different conclusion. More details regarding the number of relevant factors will be discussed in subsequent part of this chapter. Furthermore, a multicollinearity between factor two and factor three was reported.

This finding is similar to that of Pari (1980). Pari found multicollinearity among the factors and among the betas. However, these authors could not explain where this multicollinearity came from.

One possible explanation is that the idiosyncratic term is also priced. From the APT equation,

$$E_i = RP_0 + \sum_j RP_j B_{ij} \quad (2-1)$$

If the idiosyncratic error term (e_i) is also priced, then equation (2-1) will look like this

$$E_i = RP_0 + RP_1 b_{i1} + \dots + RP_k b_{ik} + RP_{k+1} e_i \quad (2-2)$$

$$\begin{aligned} \text{where } e_i &= \left(\sum_i R_i - E_i \right)^2 / n)^{1/2} \\ &= \left(\sum_i b_{ij} F_1 + \dots + b_{ik} F_k + e_i \right)^2 / n)^{1/2} \end{aligned}$$

Substitution of e_i term into (2-2) results in

$$\begin{aligned} E_i &= RP_0 + RP_i b_{ij} + \dots + RP_k b_{ik} + \\ &RP_{k+1} \left(\sum_i (b_{i1} F_1 + \dots + b_{ik} F_k + e_i)^2 / n \right)^{1/2} \end{aligned} \quad (2-3)$$

Equation (2-3) includes both diversifiable and nondiversifiable risk. In this case, multicollinearity is expected to be present in (2-2) such that inferences on RP_1 , through RP_{k+1} are misleading.

If the idiosyncratic noise term is priced, then this can mean that the APT does not hold for a large enough portfolio. However, Pari allowed the small sample size (for example, $n=5, 8, 10, 15$, etc.) in the industry portfolio and Gehr employed the randomly selected 41 individual securities instead of well-diversified portfolios. Thus whether the idiosyncratic error term (e_i) is priced or not is not completely clear, because the former has the unknown small sample properties problem and the latter has the measurement error problem.

The pioneering study of Roll and Ross (1980) uses the CRSP daily returns of 1260 securities listed on the NYSE or AMEX on both July 3, 1962, and December 31, 1972, providing a maximum of 2619 daily returns. These 1260 firms are classified into 42 groups of 30 individual securities based on alphabetical order. Maximum likelihood factor analysis (MLFA) is performed with the first group of 30 securities to determine the number of relevant factors. Roll and Ross found that there exists only a 2 percent chance for more than 5 significant factors in the data from the resulting likelihood ratio test. Thus, RR retain five factors for the remaining 41 groups. To estimate factor loadings, RR decomposed the covariance matrix into

$$V = B B' + D \quad (2-4)$$

where B = the matrix of factor loadings

Λ = the matrix of factor covariances

D = the diagonal matrix of asset variances

They assert that V will be unaltered by any transformation which leaves $B \Lambda B'$ unaltered such that if P is an orthogonal transformation matrix, $PP' = I$, then

$$\begin{aligned} V &= B \Lambda B' + D \\ &= B P P' \Lambda P P' B' + D \\ &= (B P) (P' P) (B P)' + D \end{aligned} \quad (2-4-a)$$

B is chosen from the sample covariance matrix V such that Λ is set to the identity matrix I by orthogonal factor analysis and so

$$V = B B' + D \quad (2-4-b)$$

As a consequence a cross-sectional GLS regression yields the estimates of risk premium for each day t as

$$RP_t = (B' V^{-1} B)^{-1} B' V^{-1} r_t \quad (2-5)$$

From (2-4-b) and (2-5), RR claim that the estimated risk premia are mutually independent and admit simple t -tests of significance. Using the estimated factor loadings in the second step cross-sectional GLS regression, RR conclude that at least three and possibly four "priced" factors are found in the return generating process. There exist, however, at least three potential problems.

First, RR's use of grouped data, i.e., 42 groups of 30 individual securities, causes problems associated with the determination of the number of factors with the comparison of factors among different groups. This point is further elaborated in the following part of this chapter.

Second, there exists the well-known nonuniqueness of factor loadings problem. For example,

$$\begin{aligned}
 V &= B B' + D \\
 &= (B P) (P' B') + D \\
 &= B^* B^{*'} + D
 \end{aligned} \tag{2-6}$$

It is seen from equation (2-6) that there exists an orthogonal matrix P which makes factor loadings nonunique. As a result, the estimated risk premia in equation (2-7) are also likely to contain the same kind of problem. To be more specific,

$$\begin{aligned}
 RP_t &= (B' V^{-1} B)^{-1} B' V^{-1} r_t \\
 &= (P' B' V^{-1} B P)^{-1} B' V^{-1} r_t \\
 &= (B^{*'} V^{-1} B^*)^{-1} B' V^{-1} r_t
 \end{aligned} \tag{2-7}$$

Hence, RP_t obtained from (2-5) fashion is also subject to the nonuniqueness problem.

Third, RR's utilization of t-tests for the estimated risk premia is not entirely correct because of the nonuniqueness of factor loadings.

Another interesting study using the portfolio approach is that of Reinganum (1981). By forming ten market value

portfolios, Reinganum refutes the APT because he finds that portfolios of small firms earn an average 20 percent per year more than portfolios of large firms, even after controlling for the APT risk, i.e., by measuring three-, four-, and five-factor models. If the APT is a reasonable alternative to the CAPM, it must eliminate or explain the empirical anomaly of the size effect which occurs in the CAPM. Thus, Reinganum argues that the multi-factor APT model as well as a single-factor CAPM are misspecified, because the multi-factor APT model fails to eliminate or explain the size effect.

The second approach employed in the empirical tests of the APT is the base portfolio approach or security approach proposed by Chen (1983). Chen applies this approach to a base group of 180 securities to estimate factor loadings. Assuming that the economically relevant factors are common to each security, he infers the estimate of factor loadings for the remaining groups from that of the base portfolio. However, one weakness in this base portfolio approach is that the potential estimation inefficiency can be magnified when the remaining factor loadings are inferred from the estimated factor loadings of the base portfolio.

Using the base portfolio approach, Chen finds that contrary to the finding of Reinganum (1981), firm size does not exert any additional explanatory power after the systematic risk is adjusted by the factor loading. The difference of their conclusion may come from the different statistical techniques employed. To be specific, Chen adjusts the autocorrelation in

daily returns but Reinganum does not consider the autocorrelation problem.

The third approach is the multivariate linear regression approach introduced by Jobson (1982). By assuming k linearly independent portfolios, Jobson employs a multivariate linear regression model to test the k -factor APT. The k -factor APT hypothesis is accepted if the intercept term is zero in the multivariate regression function of the $(n-k)$ returns on the k portfolios.

Assuming the existence of k linearly independent assets or portfolios with $(K \times 1)$ excess return vector r_t , Jobson writes the return generating process of the APT as follows.

$$\begin{bmatrix} r_{1t} \\ r_{2t} \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} F_t + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} \quad (2-8)$$

where the partitioning of U , B , and e_t conform to the partitioning of r_t into r_{1t} and r_{2t} , and U_t is the $(n \times 1)$ mean premium return vector.

Subtraction of the product of $B_2 B_1^{-1}$ and the first equation in (2-8) from the second equation of (2-8) leads to

$$\begin{aligned} r_{2t} = & (U_2 - B_2 B_1^{-1} U_1) + B_2 B_1^{-1} r_{1t} + (e_{2t} \\ & - B_2 B_1^{-1} e_{1t}) \end{aligned} \quad (2-9)$$

It can also be shown that $U_2 = B_2 B_1^{-1} U_1$ if the APT holds.

Jobson argued that a test of the hypothesis of a k-factor APT model can then be carried out by testing a zero intercept term in the multivariate regression r_{2t} on r_{1t} with the restriction on e_t such that e_t can be negligible. However, it is difficult to obtain K linearly independent assets or portfolios in the first place. Although, Jobson utilizes return premium vector r_t and mean premium return vector U_t , it is not easy to estimate unobservable zero-beta rate or zero-beta portfolio. Alternatively if the risk-free rate is used, there still exists the identification problem of risk-free asset. Furthermore, Jobson's restriction on error term is unreasonable because the noise term will increase rather than decrease if assets become large when the number of economic k factors are fixed.

The fourth approach is the bilinear paradigm approach introduced by Kruskal (1976) and applied to tests of the APT by Brown and Weinstein (BW) (1983). This approach is interesting because it is designed to solve the errors-in-variables problem. It will be helpful to briefly review the bilinear paradigm.

Consider the return generating process of the APT.

$$R_i = E_i + \sum_j B_{ij} F_j + e_i \quad (2-10)$$

where R_i = the random asset returns on security i

E_i = the expected return on the i-th asset

F_j = the vector of systematic factors which
commonly influence all assets.

B_{ij} = the vector of factor loadings on asset i of
factor j.

e_i = the idiosyncratic noise term associated with asset i .

$$E(e_i) = E(F_j) = 0$$

$$E(e_i | F_j) = 0 \quad \text{and}$$

$$\text{cov}(e_i, e_j) = \begin{cases} 0 & \text{for } i \neq j \\ \sigma_e^2 & \text{for } i = j \end{cases}$$

By partitioning the n securities into two disjoint groups of n_1 and n_2 securities respectively, we can rewrite equation (2-10) using matrix notation

$$\begin{aligned} [R_1 : R_2] &= A [B_1 : B_2] + [U_1 : U_2] \\ &= [A B_1 : A B_2] + [U_1 : U_2] \end{aligned} \quad (2-11)$$

where $R = [R_1 : R_2]$ is a $(t \times n)$ matrix

$B = [B_1 : B_2]$ is a $((k+1) \times n)$ matrix

$A = [RP_0 | RP + F]$ is a $(t \times (k+1))$ matrix

and the subscripts refer to the first and second subsamples respectively.

Equation (2-11) imposes the bilinear constraint that the risk-free or zero-beta rate and risk premium for the first subgroup should be same with those of the second group. An alternative to (2-11) is that the bilinear constraint is violated:

$$[R_1 : R_2] = [A_1 B_1 : A_2 B_2] + [U_1 : U_2] \quad (2-11-a)$$

If the bilinear constraint is violated, i.e., $A_1 \neq A_2$, then it

implies that the market price for two disjoint groups is different, which is obviously inconsistent with the prediction of the APT. Hence the APT will be rejected if $A_1 \neq A_2$.

To estimate the parameters in (2-11-a), BW utilize the MLE to obtain the maximized likelihood ratio, L.

$$L = \frac{1}{|\hat{D}_c| |\hat{D}_u|^{-1}} e^{-t/2} \quad (2-12)$$

where D_c and D_u refers to the ML estimates of the constrained and unconstrained diagonal error covariance matrices respectively.

It is well known that in large samples the likelihood ratio statistic (LRT)

$$-2 \ln L = t \ln \frac{|\hat{D}_c| |\hat{D}_u|^{-1}}{|\hat{D}_u|} = t \operatorname{tr} \hat{D}_c \hat{D}_u^{-1} - nt \quad (2-13)$$

is distributed as a Chi-square with $(t(K+1))$ degrees of freedom.

If the ratio of constrained to unconstrained variances gets closer to unity, then the approximation in (2-13) becomes more accurate. Equation (2-13) can be further simplified into the F statistic as

$$F = \frac{(t - \operatorname{tr} \hat{D}_c \hat{D}_u^{-1} - nt) / t(K+1)}{nt / (nt - 2t(K+1))} \quad (2-14)$$

(Appendix of BW 1983)

Using F statistics obtained by this fashion, BW test the APT for pre-determined 3-, 5-, and 7-factor models and conclude that the three factor version explains the expected returns better

than the five and seven factor models. However the results of BW are ambiguous because the probability of accepting a pre-determined k -factor APT is dependent on the number of securities (n) and the number of factors (k) used in the test.

The fifth approach is the large asset approach employed by Lehmann and Modest (LM) (1985) and Trzcinka (1986). They examine the validity of the APT based on an analysis of the returns on large cross sections of securities. To be specific, LM increase the number of securities up to 750 and use a computationally more efficient factor analysis algorithm. They found that factor models involving 750 securities provide markedly superior performance to those involving 30 or 250 securities. Lehmann et al further claim that comparatively efficient estimation procedures, such as MLFA, significantly outperform the less efficient procedures with instrumental variables and principal component analysis (PCA).

Trzcinka (1986) also employs the large asset approach by using the eigenvalue algorithm of PCA. He analyzes 865 firms listed for all twenty years having greater than 865 observations from the 1983 daily CRSP tapes to estimate the covariance matrices. Trzcinka's study is based on CR's prediction that the eigenvectors of PCA converge to factor loadings of MLFA when the number of securities is large.

Although he claims that there is no clear-cut way to select the number of factors in terms of explained variation when k is larger than one, Trzcinka provides evidence that one eigenvalue dominates the covariance matrix, indicating that one factor model may describe security pricing. He also asserts that while only

the first eigenvalue dominates the matrix, the first five eigenvalues are growing more distinct, indicating the possibility that the 5-factor model may also represent security pricing.

The Literature Review on the Number of Factors in the APT Model

A fundamental assumption in the APT is that all security returns are generated by a small number (k) of factors. This implies that the K -factor structure is stationary across assets and over time. A number of empirical tests of the APT have been devoted to the validity of this assumption.

Following RR (1980), tests of the APT have relied on factor analysis. In an effort to determine if security returns are characterized by a k -factor return generating function, Kryzanowski and To (KT) (1983) found that k is an increasing function of sample size n and a decreasing function of the number of time-series observations.

The stability of k factors as a function of sample size has also been examined by Dhrymes, Friend, and Gultekin (DFG) (1984) and DFGG (1985a) as well. Both studies found that k is an increasing function of n . Dhrymes et al (1985a) also observed that contrary to the finding by KT (1983), k is an increasing function of the number of time-series observations. However, both KT (1983) and DFGG (1985a) did not address the question of the relevant number of factors which are priced.

Roll and Ross (1984) emphasize the irrelevance of the point that factor analysis extracts more factors with large groups of securities or with larger time-series observations because

investors can diversify almost all of those redundant and irrelevant factors across industries as if the idiosyncratic disturbance term in the APT were really purely random. In short, it is the stability of k-factor APT, i.e. those that are priced, as opposed to the k-factor return generating process which is relevant.

Recently, DFGG (1985b) considered this issue of k-factor APT stability, i.e. the ability of risk measures from one period to explain returns in another period. They found that the number of significant priced factors is appreciably smaller in the second period, from March 24, 1972, to December 31, 1981, than in the first period, from July 3, 1962, to March 23, 1972 (DFGG,1985b).

In addition, they found that at least one or two factors were priced for 90 stock groups more than for the 30 stock groups, and that the number of priced factors found in the second-step cross-section regressions is much smaller than the number of factors determined in the first step factor analysis. Thus, DFGG are asserting that even the priced factors are sensitive to the sample size and the number of time-series observations. However, since the factor analytic approach has inherent problems such as nonuniqueness of factor loadings, indeterminacy of factor scores, and difficulty in determining the number of factors, DFGG conclude that their rejection of the APT is not definitive. Instead they question the empirical methodology of RR.

Focusing on this issue of the number of factors, Trzcinka (1986) finds evidence that one eigenvalue dominates the covariance matrix indicating a one factor APT model may describe

security pricing. He also claims that a five factor APT model may provide an excellent approximation of security pricing even if only one eigenvalue becomes unbounded. However, since there is no obvious way to choose the number of priced factors in terms of explained variation for values of k larger than one, it is not clear that how many factors describe security pricing.

Table 2-1 summarizes some of the important studies on the number of factors of the APT. A more detailed review of the k -factor APT is presented in Appendix A which contains statistical or theoretical methodology, and the number of factors found.

Figure 2-1 presents an overall summary of these studies.

Summary

The APT appears to be a promising testable alternative to the CAPM. Thus, in the past few years, a number of theoretical studies on the testability of the APT, empirical research on the test of the APT, and papers on the number of economically relevant or priced factors have been conducted.

This chapter, in essence, attempts to provide an overall picture of this voluminous literature of the APT. The first part of this chapter following the introduction presents the theoretical debates on the testability of the APT among Shanken (1982) and Dybvig (1983), GT (1983, 1985b), CR (1983), Ingersoll (1984), and DR (1985).

In sum, CR, DR, GT, and Ingersoll agree that the APT is always testable on subsets such that Shanken's main criticism of the untestability of the APT is generally acknowledged as false.

Table 2-1
Summary of Selected Studies on the Number of Factors of the APT

Authors	Evidence	Sample
Gehr (1978)	apparently 2, or at most 3	41 monthly stock returns
Roll and Ross (1980)	at least 3 and probably 4 "priced" factors	42 portfolios of 30 securities (CRSP daily from July 3, 1962 to Dec. 31, 1972)
Kryzanowski and To (1983)	5 factors hypothesized	11 portfolios of 50 securities and 3 portfolios of 60 securities in TSE (CRSP monthly)
Brown and Weinstein (1983)	as few as 3 economy-wide factors, and no more than 5	same as RR and 21 portfolios of 60 securities (CRSP daily)
Gibbons (1983)	6 to 9 factors including bonds	stocks and bonds of CRSP monthly from Jan. 1950 to July, 1971
Cho, Elton, and Gruber(1984)	2 factors for most of the time, occasionally 3 factors are needed	2016 securities of CRSP daily from Jan.1, 1973 to Sept.30, 1980
Dhrymes, Friend, and Gultekin (1984)	$k=f(n)$ (+)	same as RR
Cho (1984)	5 or 6 inter-group common factors (IGF) $IGF \neq f(n)$	same as RR
Gultekin and Rogalski (1985)	2 factors for bond portfolios	CRSP monthly from Jan. 1960 to Dec. 1979
DFGG (1985a)	$k=f(n,t)$ (+,+)	same as RR
DFGG (1985b)	$k_p = f(n,t)$ $X^2(RP_0) = f(n,t)$	CRSP daily of RR data from July 3, 1962 to Dec. 31, 1982
Trzcinka (1986)	1 dominant factor possibly 5 factors	865 securities of CRSP weekly

Number of authors

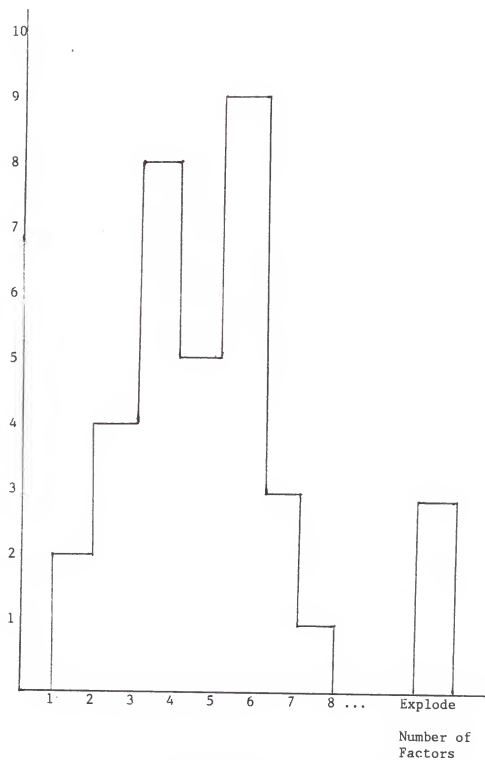


Figure 2-1
Summary of K-factor structure of the APT

The second part of this chapter presents the empirical tests of the APT by classifying these tests into 5 approaches, namely, the portfolio approach, security approach, multivariate linear regression approach, bilinear paradigm approach, and large asset approach. The pros and cons of each approach are discussed. The third part of this chapter reviews the existing literature on the number of factors in the APT model. That includes RR (1980, 1984), KT (1983), DFG (1984), DFGG (1985a, 1985b), and Trzcinka (1986) among others. A more detailed review of the k-factor APT is presented in Appendix A.

CHAPTER III
AN EXAMINATION OF THE APPROXIMATE FACTOR STRUCTURE (AFS) AND
ITS CONVERGENCE INTO THE ROSS STRICT FACTOR STRUCTURE (SFS)

Introduction

Ross (1976) introduced the arbitrage pricing theory (APT), of which the basic result is that if returns follow a factor generating model and if there exist no arbitrage opportunities, then expected returns are approximately linear in the factor loadings. Chamberlain and Rothschild (CR) (1983) have shown that with a k -factor structure in the limit economy, this result asserts the existence of k exploding eigenvalues and remaining $n-k$ bounded eigenvalues so that corresponding k eigenvectors converge and play the role of factor loadings. Hence there exists a set of $k + 1$ constants which appear as the coefficients in the linear asset pricing relation. Also, CR show that, in a limit economy, if the covariance matrix of the asset returns has only k exploding eigenvalues, then there is an AFS and it is unique.

To ensure the uniqueness of pricing coefficients, Chamberlain (1983) requires that the factor representation of the return generating process be one that is minimal so that idiosyncratic risks, including all insignificant and redundant factors, are asymptotically diversified away in the limit. However, the results from an infinite limit economy have left an

open question of whether the APT pricing should hold approximately in a finite economy. In this regard, Grinblatt and Titman (1983) and Dybvig (1983) derive the APT pricing equation within the context of a finite economy rather than an infinite limit economy.

Moreover, by deriving an explicit bound on the deviation from the APT prices across assets, they assert that the APT pricing relationship is a good approximation for the expected return of each traded security or portfolio in the finite economy as well. Connor (1984) further proves that it is impossible to empirically distinguish the finite assets from the infinite assets version of the theory.

The purpose of this chapter is to introduce the concept of the Ross SFS and the CR AFS and to examine the convergence of two factor structures as the number of assets grows. The issue of convergence of the factor structures is important because if two factor structures are divergent, then it can imply that the variance of idiosyncratic risk is unbounded, which is obviously inconsistent with the APT.

In the second part of this chapter, a detailed discussion of the Ross SFS and the CR AFS is presented. Additional characteristics of the AFS and principal component analysis (PCA) are presented next. The fourth part examines the convergence between the Ross SFS and the CR AFS by employing orthogonal procrustes rotation methodology (OPRM) and the varimax rotation methodology (VRM). In the fifth part, the sample data and the empirical results of convergence between two factor

structures are presented. In the sixth part, some evidence on the number of factors is presented. The final part summarizes the findings and makes some concluding remarks.

Factor Structure

The Ross Strict Factor Structure(SFS)

In the arbitrage pricing theory, the random returns on a subset of assets are expressed by a simple factor model

$$R = E + B F + e \quad (3-1)$$

where R = random $(n \times 1)$ column vector of asset returns

E = expected returns of R

B = $(n \times k)$ matrix of factor loadings

F = random $(k \times 1)$ vector of factors

e = random $(n \times 1)$ vector of mean-zero idiosyncratic disturbances

$$E(F) = E(e) = 0$$

$$\text{cov}(e_i, e_j) = 0 \text{ and } E(e F) = 0$$

A convenient normalization specifies that the factors are not correlated with each other, with zero mean and unit variance. The APT addresses the issue of how expected returns are formed in a capital market with no arbitrage opportunities and predicts that the expected returns of assets or portfolios are approximately linear to the factor sensitivities.

$$E = RP_0 + B RP \quad (3-2)$$

where RP_0 is a $(n \times 1)$ vector of constants representing the risk-free or zero-beta rates and RP is the $(k \times 1)$ vector of

factor risk premia. Then the following definitions are useful in understanding the Ross SFS.

Definition 1: The $(n \times n)$ covariance matrix V is defined to exhibit a strict k -factor structure if V can be additively decomposed as follows:

$$\begin{aligned} V &= B \Lambda B' + D \\ &= B B' + D \end{aligned} \quad (3-3)$$

where $V = (n \times n)$ covariance matrix of $E[(R - E)(R - E)']$,

$B = (n \times k)$ matrix of factor loadings,

$\Lambda =$ the matrix of factor covariances set identical with identity matrix I .

The diagonal elements of D remains uniformly bounded by a finite value, σ^2 in the limit when the number of assets becomes infinite.

Definition 2: Arbitrage is defined as the existence of subsequence, n_s , of arbitrage portfolios whose returns, E , satisfy

$$\begin{aligned} \lim_{n_s \rightarrow \infty} E(n_s^{-1} E) &= +\infty \\ \lim_{n_s \rightarrow \infty} \text{var}(n_s^{-1} E) &= 0 \end{aligned} \quad (3-4)$$

An arbitrage opportunity requires no net investment and produces a positive expected return with vanishing variance. Intuitively, arbitrage opportunities exist whenever there are arbitrarily large returns as the number of available asset increases.

Theorem 1: Suppose the returns on the risky investments satisfy

(3-3) and no arbitrage opportunity, and that there is a strict k -factor structure, then there exists $RP_0, RP_1, \dots, RP_k$, and A such that

$$\lim_{n \rightarrow \infty} \sum_i (E - RP_0 - \sum_j b_{ij} RP_j) < A < \infty \quad (3-5)$$

If there exists a riskless asset with return r_f , then it can be set as $RP_0 = r_f$ (for proof, see Huberman, 1982).

Theorem 1 specifies that if there is an SFS, then the absence of arbitrage opportunities implies that the mean return vector is approximately linear to the factor risk.

As the number of assets increases, the linear approximation improves and for most of the assets in a large economy, the assets' mean return vectors are almost exact linear functions of the covariances. Suppose there exist traded assets or portfolios that mimic k -factor structure, then $(RP_1 b_{i1} + \dots + RP_k b_{ik})$ can be interpreted as the risk premium on asset i . Since the rank of well-diversified portfolios is assumed to be equal to the number of factors, equation (3-5) can be interpreted as

$$\lim_{n \rightarrow \infty} \sum_i (E - RP_0 - \sum_j (E_j - RP_0) b_{ij})^2 < A < \infty \quad (3-6)$$

However, the Ross SFS is mute on the uniqueness of pricing relationship. This is because it simply identifies a set of coefficients for factor loadings whenever the choice is well-defined.

On the other hand, CR argue that, with a weaker condition than that of the Ross SFS, it is still sufficient for Ross's theorem to hold with a unique pricing relationship.

The CR Approximate Factor Structure (AFS)

Chamberlain and Roghschild (1983) emphasize that the idiosyncratic risks do not have to be independent for the APT to hold. Ingersoll (1984) also shows that the variance of residual terms needs not be diagonal. Thus, in this regard, the CR AFS is more general than the Ross SFS. A sufficient condition for the CR AFS is that there exist only k unbounded eigenvalues and that the smallest eigenvalue be positive as the number of assets becomes large. The CR AFS is given in definition 3 below.

Definition 3: The $(n \times n)$ covariance matrix V is defined to exhibit an approximate k -factor structure if there exists a sequence $\{b_{i1}, \dots, b_{ik}\}_{i=1}^n$ such that V can be additively decomposed as follows:

$$V = BB' + Q \quad (3-7)$$

where $B = (n \times k)$ matrix of factor loading

$\{Q\}$ = a sequence of positive semi-definite matrices with uniformly bounded eigenvalues for all n .

Given a symmetric matrix c , let $g_j(c)$ denote its j -th largest eigenvalue of the matrix c . Then according to definition 3, $g_1(Q)$ is finite for every n . Let $L = \sup g_1(Q)$ so that $L(1) > L(2) > \dots > L(n)$ are the eigenvalues of V and

$$L(k+1) = \sup L(k+1, n) = \sup g_{k+1} (V)$$

Chamberlain et al further propose that if V has an AFS as in (3-7), then $L(k+1)$ is finite, implying that only k largest eigenvalues explode and the remaining eigenvalues, $L(k+1)$, ..., $L(n)$ are bounded.¹ In addition, the pricing relationship from this approximate k -factor structure is unique in the following sense: suppose that there is a nested sequence of $(n \times k)$ eigenvector matrices G such that

$$V = GG' + W \quad (3-8)$$

where $\{W\}$ is a sequence of matrices with uniformly bounded eigenvalues. Then the column space of the first k eigenvectors of V and column space of B in (3-7) converge to the same value such that $GG' = BB'$ and $W = Q$ as the number of assets grows. Note that for a given V the specific decomposition in the Ross SFS is not unique because

$$\begin{aligned} V &= BB' + D \\ &= (BP)(P'B') + D \\ &= B^* B^{*'} + D \end{aligned} \quad (3-9)$$

Relationship (3-9) is easily obtainable whenever there exists any orthogonal transformation matrix P , such that $PP' = I$.

In the CR approximate k -factor structure, GG' is uniquely defined because there is no specific rotation involved. The only requirement in the CR AFS is that GG' is Gramian and W is positive semi-definite matrices whose eigenvalues are uniformly bounded for all n .²

Based on definition 3 of an approximate k-factor structure, CR proposed a theorem which is comparable with theorem 1.

Theorem 2: Suppose there is no arbitrage opportunity and the largest eigenvalue, $L(1)$, is finite according to approximate k-factor structure, then there exist $RP_0, d_1, \dots, d_k$ such that

$$\lim \sum_i (E_i - RP_0 - \sum_j d_j b_{ij}) < L(k+1)h < \infty \quad (3-10)$$

where RP_0 is the risk-free rate if a risk-free asset exists.

b_{ij} = the eigenvector corresponding to the j-th largest eigenvalue of V on asset i,

$d_j = E_j - RP_0$ for $j = 1, \dots, k$,

h = the slope of a linear trade-off between mean and standard deviation (for proof, see CR, 1983).

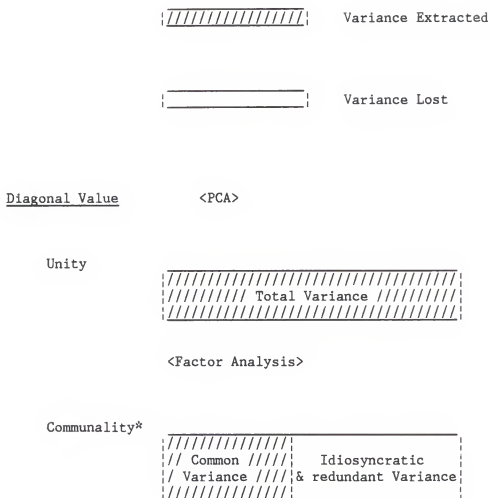
Theorem 2 specifies, similarly to theorem 1, that if there is an AFS, then the absence of arbitrage opportunities along with the boundary condition imply that the mean return vector is approximately linear to the eigenvectors. Hence, in the limit economy, the corresponding k eigenvectors converge and play the role of factor loadings. Therefore, as long as there exist traded assets or portfolios that mimic k factors, risk premium interpretation still holds for an approximate k-factor structure in the limit :

$$\lim \sum_i (E_i - RP_0 - \sum_j (E_j - RP_0) b_{ij}^2) < L(k+1)h < \infty \quad (3-11)$$

Additional Characteristics of the CR AFS and
Principal Component Analysis (PCA)

The CR (1983) AFS is built on an intuitive distinction between factor variance and idiosyncratic variance such that a well-diversified portfolio should not have idiosyncratic variance and any portfolio which is not correlated with the well-diversified portfolio contains only idiosyncratic variance. Hence, a portfolio weighting sequence which becomes well-diversified eliminates all idiosyncratic risk in the limit. This explains why PCA yields unique and consistent eigenvectors which are asymptotically equivalent to the factor loadings. While conventional factor analysis takes a diagonal matrix for a covariance matrix of idiosyncratic risk, PCA uses a scalar times an identity matrix. In this regard, Figure 3-1 will be helpful in understanding the basic difference between two methods in terms of types of variance carried into factor matrix.

Hence one is confronted with a choice between theoretically sophisticated methods, such as factor analysis and the simpler method, PCA, with a less sophisticated theoretical development but free of most of the theoretical problems of factor analysis. One can argue that factor analysis is superior to PCA because it is unacceptable that PCA extracts the same number of factors with communalities set equal to unity. For example, Lehmann and Modest (LM,1985) claim that MLFA significantly outperforms the less efficient procedures of PCA when the number of securities increases up to 750. Alternatively, one can insist that PCA is preferred because PCA, not only provides unique solutions, but also is computationally simpler and less costly in terms of



Note: * Communality is the amount of variance which an original variable shares with all other variables included in the analysis.

Figure 3-1
Types of Variance carried into factor matrix

computer time and space. For instance, CR (1983) assert that PCA provides asymptotically consistent estimates of the true factor loadings and is therefore superior to MLFA because of its computational simplicity. Recently Grinblatt and Titman (GT,1985) have shown that PCA can be used as an alternative method of forming groups of well-diversified portfolios with no idiosyncratic risk in large samples.

In any case, one can show the identity of BB' under the Ross (1976) SFS and the CR AFS if

$$D^{-1} = a I \quad (3-12)$$

where a is a scalar value and I is an $(n \times n)$ identity matrix. Relationship (3-12) is true in the limit, however, it is unrealistic to expect the exact identity for empirical finite data. However, at least, we can predict that the difference of factor patterns obtained from the Ross SFS and the CR AFS should decrease as the number of assets grows large if the APT is correct in the prediction that the variance of idiosyncratic risks should be bounded.

The following part of this chapter examines the convergence between the Ross SFS and the CR AFS by employing the OPRM and well-known VRM.

The Examination of Convergence between the Ross SFS and the CR AFS

Chamberlain et al (1983) rigorously show that if there is an AFS in a market with many assets, the corresponding k eigenvectors converge and play the role of factor loadings.

Recently GT (1985) assert that, under certain conditions, any economy satisfying the CR AFS can be transformed to satisfy the Ross SFS and converse is also true. Both studies are based on the condition of the APT pricing bound which should hold in the asset market with or without mildly correlated idiosyncratic risk if the APT holds.

Hence, investigating the convergence between the Ross SFS and the CR AFS is important because if estimated k eigenvectors are divergent from the factor loadings as the number of assets grows, it can imply that the variance of idiosyncratic risks becomes unbounded. Of course, the notion of unbounded variance is clearly inconsistent with the APT.

This issue can be directly examined by looking at the trace behavior of difference matrix product between two pattern matrices by employing the orthogonal procrustes rotation methodology (OPRM) introduced by Green (1952) and extended by Schonemann (1966), and Veliher, Peacock, and Jackson (1982), and the varimax rotation methodology (VRM).

The Orthogonal Procrustes Rotation Methodology (OPRM)

The OPRM permits a comparison where the patterns are rotated to a position of maximum similarity by determining a matrix P that will transform a given factor pattern B to a target factor pattern B^* . It is understood that such a best fit is in the least-squares sense--the sum of squares of differences between corresponding elements of BP and G is a minimum -- subject to the constraint that P is orthogonal, i.e. $PP' = I$. To illustrate the OPRM, consider the difference matrix, H , between the eigenvector

matrix, G obtained from PCA and the factor loading matrix, B estimated from MLFA

$$H = BP - G = B^* - G \quad (3-13)$$

where P is any orthogonal transformation matrix such that $PP' = I$.

Meaningful comparison between the two factor structures can then be accomplished by optimizing the congruence criterion. That can be examined by minimizing the trace of the sum of square of the difference matrix, H. Thus, the question is to find orthonormal matrices P for which the function.³

$$\begin{aligned} S_1(P_1, \dots, P_k) &= S_1 = \text{tr}(H'H) \\ &= \text{tr}(P'B'BP - 2P'B'G + G'G) \end{aligned} \quad (3-14)$$

Applying a matrix of Lagrange multipliers, M, with a side condition $PP' = P'P = I$, equation of interest can be expressed as

$$S = \text{tr}(P'B'BP - 2P'B'G + G'G) + \text{tr}[2M(PP' - I)] \quad (3-15)$$

To minimize the sum of squared errors of approximation, S is differentiated with respect to the element of P which leads to a matrix of partial derivatives

$$\frac{dS}{dP} = (B'B + B'B)P - 2B'G + 2(M + M')P = 0 \quad (3-16)$$

Rearranging equation (3-16) results in

$$B'G = [B'B + (M + M')]P \quad (3-16-a)$$

Since B and G are both assumed to have full rank, the square of matrix B'G should be positive definite.

$$B'G G'B = [B'B + (M + M')]^2 \quad (3-17)$$

Equation (3-17) can then be rewritten as

$$B'B + (M + M') = (B'G G'B)^{1/2} = Y Dy^{1/2} Y' \quad (3-18)$$

where Dy is the diagonal matrix of eigenvalues of B'G G'B. For any square symmetric matrix of rank equal to its order, such as B'G G'B, it is always possible to find an orthogonal matrix Y and a diagonal matrix Dy according to Eckart-Young theorem.⁴

Hence, P can be found from (3-16-a) and (3-18)

$$\begin{aligned} P &= (B'G G'B)^{-1/2} (B'G) \\ &= Y Dy^{-1/2} Y' B'G \end{aligned} \quad (3-19)$$

Equation (3-19) states a necessary condition on transformation matrix, P for tr (H'H) to be a minimum. To ensure the orthogonal transformation matrix P in equation (3-19) to be at minimum, tr (P'B'G) = TN should be at maximum, because

$$\begin{aligned} S_1 &= \text{tr} (H'H) \\ &= \text{tr} (P'B'B P - 2 P'B'G + G' G) \\ &= \text{tr} (B'B + G'G) - 2 \text{tr} (P'B'G) \end{aligned} \quad (3-20)$$

($\because \text{tr} (P'B'BP) = \text{tr} (B'B)$ as P is orthogonal)

where $TN = \text{tr} (P'B'G)$

$$= \text{tr} (Dy^{1/2}). \quad (3-21)$$

Proposition 1: The sufficient condition for minimization of the difference matrix between the Ross SFS and the CR AFS is found when the function $TN = \text{tr} (P'B'G)$, where P' varies without restriction, is maximized if and only if $P'B'G$ is symmetric and positive semi-definite (SPSD).

Proof. Let $P'B'G = X = U D V'$ be an Eckart-Young decomposition of $P'B'G$. Also, let any orthonormal $k \times k$ matrix A be given such that

$$\text{tr} (P'B'G) \geq \text{tr} (A'B'G) \quad (3-22)$$

Suppose $P'B'G$ is not SPSPD, then $Ux \neq Vx$ and

$$\text{tr} (P'B'G) = \text{tr} (U D V') = \text{tr} (V'U D) < \text{tr} D \quad (3-23)$$

However by taking $A = P U V'$, it can be shown that

$$\begin{aligned} \text{tr} (A'B'G) &= \text{tr} (V U' U D V') \\ &= \text{tr} D \end{aligned} \quad (3-24)$$

Obviously, (3-23) and (3-24) contradict (3-22). Therefore, $P'B'G$ is SPSPD if (3-22) holds. Conversely, let $P'B'G$ be SPSPD.

$$\text{Then} \quad P'B'G = U D V' = U D U' \quad (3-23-a)$$

$$\begin{aligned} \text{and } \text{tr} (P'B'G) &= \text{tr} (U D U') \\ &= \text{tr} (D U'U) = \text{tr} D \end{aligned} \quad (3-24-b)$$

Using the same logic, if A is an arbitrary orthonormal matrix of

order $k \times k$, then

$$\begin{aligned}
 \text{tr } (A'B'G) &= \text{tr } (A'P P'B'G) \\
 &= \text{tr } (A'P U D U') \\
 &= \text{tr } (U'A'P U D) \leq \text{tr } D
 \end{aligned} \tag{3-25}$$

Since the product $U'A'P U$ is orthonormal and has no diagonal elements greater than one, clearly (3-24-b) and (3-25) show that if $P'B'G$ is SPSP, TN is maximized. Q.E.D.

The Comparison of the Varimax Rotation Methodology (VRM) with the OPRM

Another alternative of examining the convergence of the CR AFS and the Ross SFS is the VRM introduced by Kaiser (1958) and utilized by RR (1980) for the test of the APT. The VRM also works because this method maximizes the sum of the within-factor variance of squared factor loadings in order to provide the maximum interpretability. This is probably the reason why previous studies of testing the APT almost unanimously utilize the VRM in factor analysis.

The idea is similar to that of the OPRM such that given a $(n \times k)$ matrix $B = [b_{ij}]$ of rank K , the purpose is to maximize the objective function by finding an $(k \times k)$ orthogonal matrix P so that $P'P = I$ and $B P = G$ with Kaiser's varimax criterion. This is shown as the problem of maximizing the objective function (Kaiser (1958)).

$$\text{Let } \begin{bmatrix} b_1' \\ b_j' \\ b_n' \end{bmatrix} [P_1 \ P_j \ P_k] = \begin{bmatrix} b_1 P_1 & b_1' P_i & b_1 P_k \\ b_j' P_1 & b_j' P_i & b_j' P_k \\ b_n' P_1 & b_n' P_i & b_n' P_k \end{bmatrix} \quad (3-26)$$

$$B \quad P \quad = \quad G \quad (3-26-a)$$

Also, denote the elements $b_j' P_i$ of G as g_{ji} . Then Kaiser's Varimax criterion can be written

$$\begin{aligned} \max C_1 &= n \sum_i \sum_j g_{ji}^4 - \sum_i (\sum_j g_{ji}^2)^2 \\ &= n \sum_i \sum_j (b_j' P_i)^4 - \sum_i [\sum_j (b_j' P_i)^2]^2 \\ &= C_2 + C_3 \end{aligned} \quad (3-27)$$

The first term in (3-27) can be expressed by utilizing differentiation techniques.

$$\begin{aligned} \frac{d C_2}{d P_i} &= 4 n \sum_j (b_j P_i)^3 \cdot \frac{d b_j' P_i}{d P_i} \\ &= 4 n \sum_j (b_j P_i)^3 b_j \\ &= 4 n B' (B P_i)^{(3)} \end{aligned} \quad (3-28)$$

where $B^{(k)}$ denotes for $[b_{ij}^{(k)}]$

Hence, for all k column vectors P_k in P , equation (3-28) can be written

$$\frac{d C}{d P} = 4 n B' (B P) \quad (3) \quad (3-28-a)$$

Similarly, differentiating the second term in (3-27) with respect to P_i results in

$$\begin{aligned} - \frac{d C_3}{d P_i} &= 2 \sum_j (b_j' P_i)^2 \cdot \frac{d (b_j' P_i)^2}{d P_i} \\ &= 4 \sum_j (b_j' P_i)^2 b_j' P_i \cdot \frac{d b_j' P_i}{d P_i} \\ &= 4 \sum_j (b_j' P_i)^2 b_j (b_j' P_i) \\ &= 4 B' B P_i (P_i' B' B P_i) \quad (3-29) \end{aligned}$$

$$(\because (b_j' P_i)^2 = P_i' b_j b_j' P_i$$

$$\sum_j (b_j' P_i)^2 = P_i' (\sum_j b_j b_j') P_i$$

$$= P_i' B' B P_i$$

$$b_j (b_j' P_i) = (b_j b_j') P_i)$$

Thus, for all K vectors P_k of P , equation (3-29) can generally be expressed by

$$\frac{d C_3}{d P} = -4 B' B P \text{ diag } (P' B' B P) \quad (3-29-a)$$

In addition to C_2 and C_3 , the orthogonality condition $P' P = I$ must be dealt with. Using the matrix of Lagrange multiplier M ,

$$C_4 = \text{tr} [-4 M (P' P - I)] \quad (3-30)$$

Then differentiating C_4 with regard to P is straightforward

$$\frac{d C_4}{d P} = -4 P (M + M') \quad (3-31)$$

Also, by denoting $C = C_2 + C_3 + C_4$ and differentiating C in terms of P yield the addition of equations, (3-28), (3-29-a), and (3-31),

$$\begin{aligned} \frac{d C}{d P} &= n B' (B P)^{(3)} - B' B P \text{ diag } [P' B' B P] - P (M + M') \\ &= B' n (B P)^{(3)} [\text{diag } (P' B' B P)]^{-1} [\text{diag } (P' B' B P)] \\ &\quad - B' B P \text{ diag } [P' B' B P] - P (M + M') \\ &= B' B_A D_B - B' B P D_B - P (M + M') \\ &= B' G_B D_B - B' B P D_B - P (M + M') = 0 \quad (3-32) \end{aligned}$$

with the notation of

$$\text{diag } (P' B' B P) = D_B$$

$$n(BP)^{(3)} D_B^{-1} = G_B$$

From equation (3-32), it can be seen that

$$B' G_B = B' B P + P (M + M') D_B^{-1} \quad (3-33)$$

To compare the relationship between the VRM and the OPRM, it will be useful to reiterate equation (3-16-a) of the OPRM.

$$B' G = [B' B + (M + M')] P \quad (3-16-a)$$

Equations (3-33) for the VRM and (3-16-a) for the OPRM look apparently similar in appearance so that one is tempted to investigate the possibility of interchangeability using one methodology to another. However, that is not possible, because equation (3-33) differs from equation (3-16-a) by a matrix D_B^{-1} in addition to the difference of G_B from G , where G_B is the function of P which is unknown. Hence, in general, equations (3-33) and (3-16-a) will be different, since two questions are involved in different rotation. Therefore it is of interest to compare both the OPRM and the VRM in terms of their practical usefulness.

Intuitively, the two methodologies are different in terms of their goals. Whereas the goal of the OPRM is to obtain maximum similarity between two factor structures, the purpose of the VRM is to acquire maximum interpretability of unknown common factors.

The Hypothesis for an Examination of Convergence between the SFS and AFS

The two pattern matrices of SFS and AFS can be compared with by examining the average squared difference, $AS(n)$, between the comparable loadings of two different patterns.

$$AS(n) = \text{trace} (H'H) / nk \quad (3-34)$$

Furthermore, if the APT is correct, $AS(n)$ should be getting smaller as the number of assets grows large. Thus, the hypothesis of interest is

$$\begin{aligned} H_0 &: AS(n_i) \geq AS(n_j) \\ H_A &: AS(n_i) < AS(n_j) \text{ where } i < j \end{aligned} \quad (3-35)$$

where n is the number of assets. Note that n_1 is the nested subset of n_2 which is also nested subset of n_3 so that the relationship of $n_1 < n_2 < \dots < n_N$ holds for the given time period under the economy representing k -factor structure.

Test Statistic

As Bartlett (1950) derived, the statistical significance of MLFA and PCA can be tested by calculating the certain chi-square distribution respectively.⁶ However, the average squared difference term between the comparable loadings of MLFA and PCA, $AS(n)$, is involved with the square terms of each elements (H_{ij}) in difference matrix, H . Also, the distribution of their

addition ($\sum_i \sum_j H_{ij}$) is unknown and possibly very complicated.

Henceforth, it is convenient to employ a distribution-free nonparametric test. In particular, the Wilcoxon (1945) rank sum test (or equivalently signed-rank test) is quite appropriate to test two related samples even when the assumption of the t test can not be satisfied.

To illustrate the Wilcoxon rank sum test, it will be useful to express the hypothesis (3-35) in terms of rank sum for two samples.

$$H_0 : RS \geq d(a, n_i, n_j) \quad (DF > 0) \quad \text{or}$$

$$: RS \geq [n_i (n_i + n_j + 1) - d(a, n_i, n_j)] \quad (DF = 0)$$

$$H_A : RS \leq [n_i (n_i + n_j + 1) - d(a, n_i, n_j)] \quad (3-36)$$

where RS denotes the sum of ordered rank of n_j observations when the N ($N = n_i + n_j$) observations are ordered from least to greatest. The $d(a, n_i, n_j)$ denotes the distribution of rank sum such that the constant $d(a, n_i, n_j)$ satisfies $\text{Prob} [RS \geq d(a, n_i, n_j)] = a$, and DF represents the difference between $AS(n_i)$ and $AS(n_j)$. The significance level, a , is calculated by using the Wilcoxon (1945) formula for the probability of occurrence of any given total rk or a lesser total by chance under the assumption of same population.

$$\text{Prob} = 2 \left\{ 1 + \sum_{i=1}^{rk} \sum_{j=1}^i \prod_{j=1}^i \right\}$$

$$- \sum_{n=1}^{rk-q} [(rk-q-n+1) \prod_{j=1}^{q-1} j] / \frac{2q}{q \times q} \quad (3-37)$$

where $\prod_j^i$ represents the number of j -part partitions of i ,
 rk is the rank sum of each observations, i.e., $0, 1, 2, \dots, rk$,
 q is the number of replicates, and
 n is the number of observations.

When the assumptions of the parametric t test are met, the efficiency of the Wilcoxon rank sum test compared with the t -test is 95.5 percent. For small samples, the efficiency is near 95 percent.

Empirical Results

Data

The sample data for this study consist of all the companies in the Center for Research on Security Prices (CRSP) 1984 monthly return files with no missing observations between January 1960, to December 1983. This results in a total of 452 firms. To examine the convergence issue, the sample portfolios are constructed in a way such that a nested sequence of sample covariance matrices starts with a covariance matrix of 30 securities and adds rows and columns randomly but without changing the previous rows and columns. The nested sequence selected is $n=30, 60, 90$, and 110 . Accordingly, the random replication of 8, 7, 5, and 4 portfolios with $n=30, 60, 90$, and 110 respectively can be performed without difficulty. The data are briefly described in Table 3-1. Since MLFA relies on test

Table 3-1
Data Description

Source	:	Center for Research on Security Prices Graduate School of Business University of Chicago. 1984 Monthly Returns File
Selection Criterion	:	Randomly selected 30, 60, 90, and 110 individual securities from those listed on the NYSE on both January 1960 and December 1983. The total number of firms used is 452.
Basic Data Unit	:	Return adjusted for all capital changes and including dividends
Sample time Period	:	I January 1960 - December 1983 II January 1960 - December 1971 III January 1972 - December 1983

Note: Sample Subperiod II and III include 144 months, and time period I has 288 months respectively.

statistics that assume multivariate normality, monthly data are utilized as they have been shown to be more reliable than daily return data by Fama (1976).

Empirical Results

The results of the first sample portfolio with regard to the average squared difference of factor pattern matrices of the Ross SFS and the CR AFS are presented in Tables 3-2 through 3-4. The average squared difference between comparable loadings on two different patterns of time period I, II, and III is reported when the number of assets becomes large from 30 up to 110. Panel A showed the results based on the OPRM. It is seen that for the entire sample period from 1960 to 1983, only 2 cases, when $n=90$ and $k=2$, and $n=110$ and $k=3$ have the trace observation of $AS(n_i) < AS(n_j)$ where $i < j$ (7.14 percent of rejection rate). When the time period of the sample is split into two halves, it is obvious that visual rejection of the null hypothesis, $AS(n_i) > AS(n_j)$ occurs more often than the entire sample period. To be specific, panel A of Table 3-3 shows that there are 7 cases of visual rejections (25.0 percent) and panel A of Table 3-4 indicates 4 cases of visual rejections (14.29 percent).

Panel B of Tables 3-2 up to 3-4 presented the results based on the varimax rotation methodology (VRM). Similarly as in the OPRM, it is seen that for the entire sample period, only 3 cases belong to the visual rejection. However, at this time, the elements of rejection are different, that is, when $n=90$, $k=4$, 5, and 6. For the two subperiods, there are 5 and 9 cases of visual rejections. The use of other sample portfolios exhibits, in

Table 3-2
Average Squared Difference of Factor Pattern Matrices of SFS and
AFS Expressed by $AS(n) = \text{tr}(H'H) / nk$

K	n	30	60	90	110
Panel A: Based on the Orthogonal Procrustes Rotation Methodology (OPRM)					
1		.00046	.00037	.000073	.000055
2		.00093	.00031	.00058*	.00023
3		.00138	.00065	.00020	.00024*
4		.00174	.00063	.00028	.00024
5		.00969	.00106	.00066	.00043
6		.00295	.00198	.00141	.00092
7		.0242	.00385	.00123	.00122*
Panel B: Based on the Varimax Rotation Methodology (VRM)					
1		.00046	.00037	.000073	.000054
2		.20188	.19481	.16161	.15049
3		.17373	.15650	.12850	.11982
4		.11903	.10379	.10684*	.10037
5		.11247	.09837	.10181*	.09925
6		.111958	.07628	.09373*	.09284
7		.107213	.08568	.08067	.08002

Note: Time period is Jan. 1960 - Dec. 1983.

* denotes the case when the observation of $AS(n_j) > AS(n_i)$

When $k=1$, rotation is not possible. Thus, the unrotated differences are reported.

Table 3-3
Average Squared Difference of Factor Pattern Matrices of SFS and
AFS Expressed by $AS(n) = \text{tr}(H'H) / nk$

K	n	30	60	90	110
Panel A : Based on the OPRM					
1		.00040	.00017	.000061	.000051
2		.00085	.00034	.00024	.00023
3		.00185	.00089	.00118*	.00132*
4		.00209	.00092	.00169*	.00114
5		.00565	.00446	.00125	.00284*
6		.00264	.00342*	.00097	.00198*
7		.00744	.00125	.00168	.00147*
Panel B : Based on the VRM					
1		.00040	.00017	.000061	.000050
2		.021335	.19406	.16926	.11904
3		.22101	.13224	.16557*	.13478
4		.16107	.10241	.14884*	.13944
5		.10728	.09648	.10352*	.11463*
6		.10456	.08850	.10375*	.10247*
7		.009814	.08547	.07881	.08539*

Note: Time period is Jan. 1960 - Dec. 1971.

Table 3-4
Average Squared Difference of Factor Pattern Matrices of SFS and
AFS Expressed by $AS(n) = \text{tr} (H'H)/nk$

K	n	30	60	90	110
Panel A : Based on the OPRM					
1		.00057	.00089*	.00013	.00009
2		.00115	.00046	.00179*	.00144*
3		.00163	.00121	.00019	.00023*
4		.00238	.00082	.00028	.00025
5		.00250	.00076	.00061	.00032
6		.01830	.00211	.00111	.00068
7		.00282	.00107	.00148*	.00164*
Panel B: Based on the VRM					
1		.00057	.00089*	.0013	.00094
2		.22740	.23304*	.21252	.20158
3		.12537	.12942*	.15514*	.13956
4		.13714	.11762	.12600*	.11106
5		.12003	.09595	.10943*	.10273*
6		.10811	.09553	.09606*	.09824*
7		.09449	.08662	.09387*	.08995

Note: Time period is Jan. 1972 - Dec. 1983.

Table 3-5
a-values based on Wilcoxon's Two-Sample Rank-Sum Test
using the OPRM

K	n	30 vs. 60	60 vs. 90	90 vs. 110
Panel A : based on Jan. 1960 - Dec. 1983				
1		.007***	.053*	.095*
2		.007***	.037**	.056*
3		.010***	.074*	.095*
4		.014**	.053*	.143
5		.020**	.134	.206
6		.047**	.216	.206
7		.076*	.265	.365
Panel B: based on Jan. 1960 - Dec. 1971				
1		.020**	.134	.278
2		.014**	.172	.206
3		.027**	.216	.278
4		.036**	.265	.278
5		.047**	.216	.365
6		.116	.134	.206
7		.140	.265	.452
Panel C : based on Jan. 1972 - Dec. 1983				
1		.005***	.037**	.056*
2		.007***	.024**	.032**
3		.007***	.053*	.056*
4		.010***	.074*	.095*
5		.014**	.074*	.095*
6		.060*	.134	.143
7		.095*	.172	.143

Note: * significant at $\alpha = .10$ level
 ** significant at $\alpha = .05$ level
 *** significant at $\alpha = .01$ level

The number in parentheses represents the a-value of $AS(n_i) \geq AS(n_j)$ instead of $AS(n_i) < AS(n_j)$ where $i < j$

Table 3-6
a-values based on Wilcoxon's Two-sample Rank-Sum Test using VRM

K	n	30 vs. 60	60 vs. 90	90 vs. 110
Panel A : based on Jan. 1960 - Dec. 1983				
1		.007***	.053*	.095*
2		.076*	.134	.206
3		.060*	.101	.143
4		.047**	.101	.206
5		.036**	.074*	.095*
6		.095*	.134	.206
7		.060*	.101	.143
Panel B : based on Jan. 1960 - Dec. 1971				
1		.020**	.134	.278
2		.095*	.216	.365
3		.060*	.172	.206
4		.076*	.134	.143
5		.060*	.101	.143
6		.168	.216	.365
7		.116	.134	.278
Panel C : based on Jan. 1972 - Dec. 1983				
1		.005***	.037**	.056*
2		.047**	.101	.143
3		.060*	.074*	.095*
4		.036**	.053*	.056*
5		.027**	.037**	.032**
6		.076*	.101	.143
7		.047**	.074*	.102

Note: * significant at $\alpha = .10$ level
 ** significant at $\alpha = .05$ level
 *** significant at $\alpha = .01$ level

The numbers in parentheses represent the a-value of $AS(n_i) \geq AS(n_j)$ instead of $AS(n_i) < AS(n_j)$ where $i < j$.

general, similar behavior in terms of the magnitude of the average squared difference and in terms of the number of visual rejections for both cases of the OPRM and the VRM.

To apply the Wilcoxon (1945) rank sum test, it is necessary to replicate the experiments in order to make sure random drawings for each k .⁷ Under the given sample of 452 securities, it is possible to replicate 4 portfolios with $n=110$, 5 portfolios with $n=90$, 7 portfolios with $n=60$, and 8 portfolios with $n=30$. Although, 15 portfolios can be constructed at maximum for the case of $n=30$, 8 portfolios are adopted since 8 replications are enough for our purpose.

Table 3-5 presents the α -values based on Wilcoxon's two sample rank-sum test using the OPRM including the entire sample period and two subperiods. The α -value denotes the probability that the rank-sum of the n_i is equal to or less than the smaller rank-sum and the chance that the rank-sum of the n_i is equal to or greater than the greater rank-sum between the two samples, n_i and n_j .

Panel A showed the results based on the entire sample period. It is seen that when $n=30$ vs. 60, the average squared difference is significant at 1 percent level in the case of $k=1$ through 3, significant at 5 percent level in the case of $k=4$ through 6, and significant at 10 percent level in the case of $k=7$. When $n=60$ vs. 90, it is less obvious because only one case of $k=2$ is significant at 5 percent level, other cases of $k=1, 3$, and 4 are significant at 10 percent level and other three cases are insignificant. When $n=90$ vs. 110, it is seen that the average squared difference is significant, indicating the possible

convergence of two factor structures. Therefore, obviously, it seems that the average squared difference of two factor structures using the entire sample period is getting smaller when the number of assets becomes large. This finding is consistent with the prediction of the APT. In the comparison of $n=30$ vs. 60, all cases are significant, implying that the average squared differences are significantly larger than those of 60 securities. This finding may imply that RR's formulation of portfolios consisted of 30 securities each is not efficient in the empirical testing of the APT.

Panel B presented the results based on the first half period from January 1960, to December 1971. The results of first subperiod is quite different from those of the entire period. When $n=60$ vs. 90 and 90 vs. 110, no cases are significant, indicating the earlier convergence than those of the entire period. Even in $n=30$ vs. 60, 5 cases of $k=1$ up to 5 are significant at 5 percent level, implying the magnitude of average squared differences is smaller than those of the entire period.

Panel C provided the results based on the second half period from January 1972, to December 1983. When $n=60$ vs. 90 and $n=90$ vs. 110, five cases of $k=1$ up to 5 are significant at 10 percent level indicating the weak tendency of convergence (possibly ambiguous). When $n=30$ vs. 60, the average squared difference is significant at 1 percent level in the case of $k=1$ through 4, significant at 5 percent when $k=5$, and significant at 10 percent level when $k=6$ and 7.

In Table 3-6, the α values based on Wilcoxon's rank-sum test using the VRM for the two subperiods and entire period are presented. The empirical results using the VRM are not completely consistent with those using the OPRM. Specifically, panel A showed that the only two cases each of $n=60$ vs. 90 and $n=90$ vs. 110 are significant and all other remaining values are insignificant, implying the possible earlier convergence of two factor structures.

However, Panel B provides the results that when $n=60$ vs. 90 and $n=90$ vs. 110, no case is significant at 10 percent level. Thus, in this period, whether the two factor structures are converging seems more obvious.

Panel C showed the results based on the second half period. It is seen that this subperiod has more significant cases than the first subperiod. Although the results using the VRM are not perfectly consistent with those using the the OPRM, those are generally supportive of convergence.

In sum, it was found that, by utilizing the OPRM (and by the VRM also), the average squared difference of the two factor loadings obtained from SFS and AFS is getting smaller when the number of assets grows, indicating the convergence of two factor structures. This finding is consistent with the APT prediction. Hence, these tests could not reject the null hypothesis that the variance of idiosyncratic risks is bounded. Therefore, it was not possible to reject one of the important APT implication that idiosyncratic noise terms represent diversifiable risk and thus should not affect asset pricing.

Some Evidence on the Number of Factors

This section provides a closer look into the empirical results of both MLFA and PCA. Specifically, in Tables 3-8 through 3-19, the results of MLFA and PCA for 30, 60, 90, and 120 securities, covering two subperiods and entire sample period are presented. Each table reports the results on the number of factors using MLFA, communality estimates obtained from PCA and MLFA, and squared canonical correlations based on MLFA.

To be specific, first, the results on the number of factors include those of chi-square value, probability that k factors are sufficient, Akaike's information criterion (AIC), Schwarz's bayesian criterion (SBC), and Tucker and Lewis's reliability coefficient (TLRC). As indicated by DFG (1984), DFGG (1985a), and RR (1984), the chi-square test produces an exorbitant number of factors which include priced factors as well as redundant factors or unpriced factors. AIC is acknowledged as a general criterion to estimate the number of factors when MLFA is utilized. According to AIC, the number of factors that yields the smallest value of AIC is considered best. However, AIC, like the chi-square test, tends to include factors that are statistically significant but inconsequential for practical purpose. Thus, AIC is not significantly better than the chi-square test in this respect.

SBC is a standard criterion for determining the best number of factors. Similarly to AIC, the number of factors that yields the smallest value of SBC is considered best. In general, however, SBC is established to be far less inclined to include redundant factors and irrelevant factors than either the chi-

square test or AIC. Table 3-7 summarizes the findings of Tables 3-8 through 3-19. It is seen that when the number of assets increases, the number of factors obtained from the chi-square test and AIC is also increasing, which is consistent with the findings of DFG (1984). However, what is to be investigated is the simple k-factor structure which contains only the market wide factors, not the irrelevant and redundant factors.

Panel A of Table 3-7 could provide the preliminary evidence regarding the simple k-factor structure. The range of the number of factors obtained from utilizing SBC is 2 - 4, which is similar to the finding of RR (1980). RR found evidence consistent with the APT and concluded that only about three or four priced factors are relevant in explaining returns. However, this finding is only preliminary, because SBC itself, like the second-step of RR, could not identify whether the factors are priced or not.

Another interesting finding is that the number of factors for the different time period is not completely constant. For example, the number of factors is 3 to 4 for entire period, 2 for the first half period, and 2 to 4 for the second half period.

Nevertheless, as Trzcinka (1986) also points out, whether the number of economic, that is, priced factors is constant or variable remains to be discovered by large scale pricing tests. Also, Tables 3-8 through 3-19 report Tucker and Lewis's reliability coefficient (TLRC). While MLFA is quite profitable in establishing latent traits which account for essential interrelations among observations in a domain of phenomena, MLFA

will not represent the phenomena exactly for a population of objects. In this regard, to examine the level of congruence, a reliability coefficient is defined as

$$RC = \frac{V_k}{V_k + DV_k} \quad (3-27)$$

Where RC is a reliability coefficient

V_k is a variance associated with a model having K common factors, and

DV_k is a variance representing the deviation of the model from actuality.

Tucker and Lewis (1973) estimate RC as

$$RC = \frac{M_o - M_k}{M_o - 1/n'_k} \quad (3-37-a)$$

Where M_o is the mean square for a model having zero common factors

M_k is a mean square corresponding Joreskog's function

$$F_k = \sum_{j=1}^{n-1} \sum_{j'=j+1}^n g_{kjj'}^2$$

(See Joreskog 1969).

Thus TLRC may be interpreted as indicating how well a k-factor model represents the covariance among the attributes for a population of objects.

Hence, the TLRC could indicate a very close representation and be a better indication as to whether to accept or reject the MLFA solution. Intuitively, lack of fit by TLRC would indicate

that the relations among the attributes are more complex than can be represented by k common factors. Tables 3-8 through 3-19 showed that when the number of asset increases, TLRC tends to decrease, while when the number of time observations increases, TLRC seems not much changing, and when the number of factors increase, TLRC is absolutely increasing. Thus, TLRC is a positive function of the number of factors and a negative function of the number of assets. The range of TLRC is .503-.971 when k is 1 to 7, indicating the reasonable explanatory power of MLFA.

Tables 3-8 through 3-19 report communality estimates obtained from using PCA and MLFA. The communality is given by the sum of squared factor loadings.

$$h_j^2 = \sum_{i=1}^k b_{ji}^2$$

where h_j^2 is the communality of a variable j , and

b_j is the factor loadings.

Communalities show the amount of variance that is accounted for by the k -factors taken together. These figures are useful because they show how much variance in a particular variable is accounted for by the factor solution. Intuitively, large communalities indicate that a large amount of the variance in a variable has been extracted by the factor solution and small communalities show that a substantial portion of the variance in a variable is unaccounted for by the factor solution.

The communality estimates reported in Tables 3-8 through 3-19 are final communality estimates, which are the squared multiple correlations for predicting the variables from the estimated factors. They refer to the variance explained by the factors collectively. They are obtained by taking the sum of squares of each row of factor pattern and a weighted sum of squares when variable weights are used. Specifically, the hypothetical population correlation matrix is proxied using k factors. For the weighted communality estimates of MLFA, weighting scheme is employed by using the reciprocal of its uniqueness.

The percentage of variance explained by the factor is the trace of proxied hypothetical population correlation matrix divided by the trace of the sample correlation matrix. For example, in the three factor solution of Table 3-8, an estimated 14.31 percent of the total variance when using PCA (24.37 percent when using MLFA for the weighted communality estimates, and 12.66 percent when using MLFA for the unweighted communality estimates) is systematic. Of that systematic portion, the first factor accounts for 71.56 percent and the second factor explains 17.33 percent with the third accounting for the remaining 11.11 percent. It can be seen that, from the Tables 3-8 through 3-19, when the number of assets increase, the communality estimates obtained from using PCA and MLFA increase. Also, when the number of time observations increases, the communality estimates decrease. Therefore when the number of assets increases, the explained proportion of the total variance by the factors

increases and when the number of time observation increases, the explained proportion decreases.

Tables 3-8 through 3-19 also report the squared canonical correlations obtained from using MLFA. The squared canonical correlations for each factor with the variables are measures of the extent to which the factors can be predicted from the variables and can be interpreted as the squared multiple correlation. For instance, in the three factor solution of Table 3-10, the first factor can be predicted with great accuracy ($R^2 = 95.54$ percent).

Also, in the seven factor solutions of Tables 3-8, and Table 3-9 and in the five and seven factor solutions of Table 3-18, the perfect correlations occurs because of Heywood case, which refer to the situation when a communality equals 1. Possible causes of Heywood case may include bad communality estimates, too many or too small factors extracted, and not enough data to provide the stable estimates, etc. In this paper, the Heywood case disappears, in general, when we increase the sample size and time observation.

The squared canonical correlation also showed the similar tendency to the communality estimates such that the squared canonical correlation is the positive function of the sample size, but the negative function of the time observation.

Overall, the evidence of this section is summarized as follows. First, from the results of chi-square test and AIC, the number of factors found is the positive function of the sample size, which is consistent with that of DFG (1984). However, this finding is not meaningful because chi-square test and AIC produce

Table 3-7
Summary Results of K-factor APT based on MLFA

Time Period	n	30	60	90	120
Panel A : the number of factors based on SBC					
I		3	3	3	4
II		2	2	2	2
III		2	3	4	4
Panel B : the number of factors based on AIC and chi-square test					
I		≥ 7	≥ 7	≥ 7	≥ 7
II		5(4)	≥ 7	6(≥ 7)	≥ 7
III		5	≥ 7	≥ 7	≥ 7

Note: Time period I is Jan. 1960 - Dec. 1983
 Time period II is Jan. 1960 - Dec. 1971
 Time period III is Jan. 1972 - Dec. 1983

The numbers in parentheses represent the number of factors obtained from the chi-square test, of which result is different from that of AIC.

≥ 7 denotes the case that the factors may be larger than or equal to 7.

Table 3-8
Summary Information

Number of Securities : 30
Number of Observations : 288
Sample Period : 01/60 - 12/83

The Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion (AIC)	Schwarz's Bayesian Criterion (SBC)	Tucker & Lewis's Reliability Coefficient (TLRC)
1	1276.24	.0001	1454.15	836.96	.717
2	733.64	.0001	946.79	636.39	.875
3	577.82	.0001	840.98	634.77**	.913
4	472.92	.0001	785.99	656.73	.938
5	407.21	.0001	769.84	696.27	.950
6	347.35	.0010	757.55	753.92	.962
7	300.07	.0105	756.31*	779.25	.971

Note: * represents the smallest AIC and

** represents the smallest SBC for Tables 3-8 through 3-19
respectively.

Table 3-8 (continued)
Communality Estimates

K	Total Estimated Communality (%)	Percentage of Total Estimated Communality Attributed to Each Factor						
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Panel A : based on PCA

		1	2	3	4	5	6	7
1	10.24	100						
2	12.72	80.50	19.50					
3	14.31	71.56	17.33	11.11				
4	15.60	65.64	15.90	10.19	8.27			
5	16.60	61.69	14.94	9.58	7.71	6.08		
6	17.54	58.38	14.14	9.06	7.30	5.74	5.54	
7	18.46	55.47	13.43	8.16	6.93	5.46	5.25	4.85

Panel B : based on MLFA

1	14.45 (9.56)	100						
2	21.83 (11.62)	80.67	19.33					
3	24.37 (12.66)	75.36	17.90	6.74				
4	26.63 (13.43)	71.54	19.05	6.57	4.84			
5	29.12 (13.98)	68.65	16.28	6.77	4.89	3.41		
6	30.97 (14.53)	35.62	21.54	15.92	10.82	10.96	5.14	
7	30.70 (15.26)	35.96	20.36	14.01	11.89	11.17	2.02	1.58

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables							
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	1	2	3	4	5	6	7
1	93.53						
2	94.63	80.84					
3	94.84	81.36	62.13				
4	95.01	81.95	63.65	56.29			
5	95.24	82.57	66.29	58.75	50.10		
6	95.36	82.98	68.01	59.22	51.31	47.32	
7+	100.00	93.99	78.80	68.46	59.13	50.97	48.01

Note: The numbers in parentheses in Panel B for Tables 3-8 through 3-19 represent unweighted communality estimates.

+ Indicates a Heywood case for Tables 3-8 through 3-19.

Table 3-9
Summary Information

Number of Securities : 30
Number of Observations : 144
Sample Period : 01/60 - 12/71

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	756.74	.0001	948.67	563.43	.752
2	513.12	.0001	742.76	603.53**	.896
3	421.21	.0043	699.97	523.72	.940
4	358.79	.0718	686.97	557.31	.966
5	308.10	.2882	684.37*	594.62	.987
6	266.11	.5555	688.98	634.05	1.004+
7	232.31	.7254	700.37	675.38	1.016+

Table 3-9 (continued)
Communality Estimates

K	Total Estimated Communality (%)	Percentage of Total Estimated Communality Attributed to Each Factor						
Panel A : based on PCA								
		1	2	3	4	5	6	7
1	10.33	100						
2	12.87	80.26	19.74					
3	14.44	71.54	17.59	10.87				
4	15.70	65.80	16.18	10.00	8.02			
5	16.86	61.27	15.07	9.31	7.47	6.88		
6	17.92	57.65	14.17	8.76	7.03	6.47	5.89	
7	18.88	54.71	13.45	8.32	6.67	6.14	5.59	5.11

Panel B : based on MLFA								
1	14.74(9.67)	100						
2	20.99(11.75)	81.09	18.91					
3	24.55(12.84)	74.13	17.19	8.68				
4	27.79(13.68)	69.59	16.08	7.77	6.56			
5	31.43(14.40)	65.64	14.83	9.64	5.73	4.16		
6	33.60(15.08)	63.69	14.38	8.15	5.45	4.46	3.87	
7	32.87(15.74)	31.88	20.20	17.92	11.86	5.09	8.85	3.44

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	93.65						
2	94.45	79.88					
3	94.79	80.84	68.01				
4	95.08	81.71	68.40	64.57			
5	95.38	82.32	75.18	64.25	56.76		
6	95.54	82.84	73.23	64.67	60.06	56.71	
7+	100.00	93.79	83.25	66.45	62.92	58.20	52.30

Table 3-10
Summary Information

Number of Securities : 30
Number of Observations : 144
Sample Period : 01/72 - 12/83

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	938.64	.0001	1147.86	663.03	.675
2	604.83	.0001	843.70	554.01**	.850
3	479.93	.0001	764.93	556.20	.906
4	389.16	.0054	720.74	574.20	.948
5	330.04	.0783	708.89*	606.88	.971
6	285.50	.2473	710.77	644.94	.986
7	245.31	.5005	715.05	682.72	1.000

Table 3-10 (continued)
Communality Estimates

K	Total Estimated Communality (%)	Percentage of Total Estimated Communality Attributed to Each Factor						
Panel A : based on PCA								
		1	2	3	4	5	6	7
1	10.24	100						
2	13.28	79.37	20.63					
3	15.18	69.43	17.98	12.59				
4	16.61	63.46	16.44	11.44	8.56			
5	17.75	59.38	15.38	10.70	8.06	6.42		
6	18.76	56.18	14.55	10.13	7.62	6.08	5.38	
7	19.75	53.37	13.82	9.62	7.23	5.77	5.11	5.03

Panel B : based on MLFA								
1	45.65(9.89)	100						
2	25.67(12.23)	79.51	20.49					
3	29.58(13.64)	72.38	18.90	8.72				
4	33.80(14.64)	67.81	17.69	8.20	6.30			
5	37.04(15.42)	64.28	17.04	8.67	6.18	3.83		
6	40.36(16.06)	61.77	16.75	8.45	6.14	3.89	3.02	
7	43.47(16.71)	59.74	16.08	8.10	6.12	3.68	3.27	3.01

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	93						
2	95.33	84.04					
3	95.54	84.83	72.11				
4	95.82	85.68	73.46	68.08			
5	95.97	86.31	76.26	69.61	58.75		
6	96.14	87.11	77.30	71.25	61.10	55.00	
7+	96.29	87.49	77.90	72.66	61.57	58.61	56.55

Table 3-11
Summary Information

Number of Securities : 60
Number of Observations : 288
Sample Period : 01/60 - 12/83

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	3956.04	.0001	4531.30	2485.43	.6801
2	2767.22	.0001	3367.29	2011.48	.8353
3	2358.39	.0001	3045.17	1956.64**	.8830
4	2113.01	.0001	2897.47	1987.19	.9085
5	1940.85	.0001	2826.69	2054.37	.9242
6	1815.16	.0001	2804.03	2143.76	.9333
7	1685.99	.0001	2774.85*	2328.07	.9440

Table 3-11 (continued)
Communality Estimates

K	Total Estimated Communality (%)	Percentage of Total Estimated Communality Attributed to Each Factor						
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Panel A : based on PCA

		1	2	3	4	5	6	7
1	19.11	100						
2	23.42	81.61	18.40					
3	25.78	74.13	16.72	9.15				
4	27.60	69.24	15.62	8.55	6.59			
5	29.14	65.58	14.79	8.10	6.25	5.28		
6	30.45	62.76	14.15	7.75	5.98	5.06	4.30	
7	31.70	60.28	13.60	7.44	5.74	4.86	4.13	3.97

Panel B : based on MLFA

1	27.87(18.39)	100						
2	40.21(22.27)	81.67	18.33					
3	45.48(24.12)	75.55	17.19	7.26				
4	49.28(25.42)	71.94	16.29	7.24	4.53			
5	52.50(26.43)	69.43	15.71	7.03	4.52	3.31		
6	55.61(27.18)	67.61	15.36	6.83	4.37	3.22	2.61	
7	58.76(27.97)	65.66	14.98	6.84	4.39	3.15	2.54	2.46

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	96.54						
2	97.05	88.06					
3	97.17	88.66	76.75				
4	97.26	88.93	78.13	68.99			
5	97.33	89.19	78.66	70.38	63.51		
6	97.41	89.52	79.17	70.84	64.15	59.13	
7+	97.47	89.79	80.07	72.07	64.90	59.86	59.08

Table 3-12
Summary Information

Number of Securities : 60
Number of Observations : 144
Sample Period : 01/60 - 12/71

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	2582.08	.0001	3300.24	1828.31	.7143
2	2078.30	.0001	2834.76	1683.18**	.8550
3	1932.66	.0001	2789.98	1746.91	.8805
4	1787.83	.0001	2742.37	1807.75	.9081
5	1686.07	.0001	2743.15	1891.29	.9220
6	1584.99	.0018	2741.50	1972.14	.9371
7	1486.59	.0154	2739.86*	2051.50	.9528

Table 3-12 (continued)
Communality Estimates

K	Total Estimated Communality (%)	Percentage of Total Estimated Communality Attributed to Each Factor						
Panel A : based on PCA								
		1	2	3	4	5	6	7
1	19.15	100						
2	23.30	82.19	17.81					
3	25.36	75.51	16.36	8.13				
4	27.27	70.22	15.22	7.55	7.01			
5	28.94	66.17	14.34	7.12	6.60	5.74		
6	30.55	62.68	13.58	6.74	6.25	5.43	5.27	
7	32.04	59.77	12.95	6.43	5.96	5.18	5.02	4.65
Panel B : based on MLFA								
1	27.82(18.46)	100						
2	38.90(22.13)	82.06	17.84					
3	43.08(23.64)	77.16	16.69	6.15				
4	47.65(25.07)	72.91	15.97	5.73	5.38			
5	51.47(26.16)	69.98	15.33	5.63	5.10	3.95		
6	55.33(27.27)	67.25	14.91	5.40	5.11	3.77	3.56	
7	59.21(28.41)	64.85	14.25	5.49	4.91	3.76	3.39	3.34

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	96.53						
2	96.97	87.41					
3	97.08	87.80	72.52				
4	97.20	88.39	73.17	71.98			
5	97.30	88.75	74.35	72.42	67.06		
6	97.38	89.19	74.94	73.89	67.62	66.24	
7+	97.46	89.41	76.47	74.43	69.01	66.77	66.43

Table 3-13
Summary Information

Number of Securities : 60
Number of Observations : 144
Sample Period : 01/60 - 12/71

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	3270.63	.0001	4116.30	2236.34	.5962
2	2532.15	.0001	3375.62	1953.61	.7639
3	2217.91	.0001	3131.80	1917.82**	.8264
4	2006.74	.0001	3006.16	1939.64	.8644
5	1839.81	.0001	2929.45	1984.44	.8924
6	1718.62	.0001	2904.34	2053.56	.9088
7	1607.82	.0001	2888.43*	2125.79	.9236

Table 3-13 (continued)
Communality Estimates

K	Total Estimated Communality (%)	Percentage of Total Estimated Communality Attributed to Each Factor						
Panel A : based on PCA								
		1	2	3	4	5	6	7
1	19.73	100						
2	24.89	79.27	20.73					
3	27.78	71.02	18.57	10.41				
4	30.02	65.72	17.19	9.62	7.47			
5	31.97	61.71	16.14	9.03	7.01	6.07		
6	33.54	58.83	15.38	8.61	6.68	5.78	4.70	
7	34.99	56.39	14.75	8.25	6.40	5.55	4.50	4.15
Panel B : based on MLFA								
1	30.68(18.97)	100						
2	47.32(23.78)	79.53	20.41					
3	55.69(26.23)	71.50	19.64	8.86				
4	62.13(28.05)	67.02	18.65	8.51	5.79			
5	67.75(29.55)	64.22	17.80	8.15	5.64	4.19		
6	73.01(30.66)	62.07	17.20	8.01	5.57	4.13	3.02	
7	77.81(31.75)	60.28	16.73	7.71	5.59	4.01	2.94	2.75

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	96.84						
2	97.41	90.62					
3	97.55	91.62	83.12				
4	97.65	92.06	84.11	78.28			
5	97.75	92.34	84.66	79.26	73.96		
6	97.84	92.63	85.39	80.27	75.09	68.71	
7+	97.91	92.86	84.71	81.30	75.72	69.57	68.19

Table 3-14
Summary Information

Number of Securities : 90
Number of Observations : 288
Sample Period : 01/60 - 12/83

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	7711.49	.0001	9052.40	4855.87	.6599
2	6148.01	.0001	7486.17	4235.75	.7871
3	5206.58	.0001	6613.66	3960.67**	.8622
4	4764.61	.0001	6301.05	3963.70	.8930
5	4519.73	.0001	6208.38	4074.87	.9061
6	4317.68	.0001	6161.23	4206.98	.9156
7	4126.97	.0001	6123.91*	4342.16	.9245

Table 3-14 (continued)
Communality Estimates

K	Total Estimated Communality (%)	Percentage of Total Estimated Communality Attributed to Each Factor						
Panel A : based on PCA								
		1	2	3	4	5	6	7
1	28.35	100						
2	33.74	84.02	15.98					
3	37.65	75.30	14.32	10.38				
4	40.20	70.52	13.41	9.73	6.34			
5	41.98	67.53	12.84	9.31	6.07	4.24		
6	43.52	65.14	12.39	8.98	5.86	4.09	3.54	
7	45.01	62.99	11.98	8.69	5.67	3.95	3.44	3.29

Panel B : based on MLFA								
1	42.22(27.67)	100						
2	57.85(32.57)	82.75	17.25					
3	67.82(36.04)	74.48	15.56	9.96				
4	73.29(38.05)	70.96	14.77	9.63	4.63			
5	77.30(39.27)	69.04	14.23	9.31	4.51	3.91		
6	81.20(40.27)	67.52	13.84	9.10	4.42	2.82	2.29	
7	84.96(41.28)	65.93	13.45	9.02	4.33	2.78	2.28	2.20

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	97.68						
2	97.95	90.89					
3	98.06	91.35	87.10				
4	98.11	91.54	87.59	77.23			
5	98.16	91.67	87.80	77.72	69.24		
6	98.21	91.83	88.08	78.20	69.58	65.08	
7+	98.25	91.95	88.46	78.61	70.28	66.03	65.17

Table 3-15
Summary Information

Number of Securities : 90
Number of Observations : 144
Sample Period : 01/60 - 12/71

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	5460.81	.0001	7412.52	3973.54	.6785
2	4803.41	.0001	6791.81	3795.34**	.7899
3	4602.57	.0001	6730.07	3895.15	.8117
4	4407.01	.0001	6683.52	4001.06	.8314
5	4208.06	.0001	6627.78	4100.89	.8531
6	4031.17	.0001	6596.61*	4211.52	.8710
7	3890.05	.0001	6608.86	4342.38	.8815

Table 3-15 (continued)
Communality Estimates

K	Total Estimated Communality (%)		Percentage of Total Estimated Communality Attributed to Each Factor						
Panel A : based on PCA									
			1	2	3	4	5	6	7
1	28.60	100							
2	33.84	84.52	15.48						
3	36.36	78.66	14.41	6.93					
4	38.75	73.81	13.52	6.50	6.17				
5	40.99	69.77	12.78	6.15	5.83	5.47			
6	43.02	66.48	12.18	5.86	5.56	5.21	4.72		
7	44.82	63.81	11.69	5.62	5.33	5.00	4.53	3.99	

Panel B : based on MLFA								
1	43.42(27.93)	100						
2	57.28(32.66)	84.11	15.89					
3	62.62(34.61)	79.70	14.96	5.34				
4	68.22(36.40)	75.87	14.17	5.31	4.65			
5	74.08(38.16)	72.52	13.67	5.10	4.36	4.35		
6	79.95(39.76)	69.69	13.02	5.08	4.42	4.22	3.58	
7	84.71(41.07)	67.68	12.69	4.92	4.31	4.08	3.48	2.83

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	97.75						
2	97.97	90.10					
3	98.04	90.36	76.98				
4	98.10	90.63	78.34	76.02			
5	98.17	91.01	79.07	76.39	76.29		
6	98.24	91.24	80.23	77.92	77.10	74.07	
7+	98.29	91.49	80.65	78.48	77.60	74.67	70.58

Table 3-16
Summary Information

Number of Securities : 90
Number of Observations : 144
Sample Period : 01/72 - 12/83

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	6709.07	.0001	9024.63	4779.60	.5454
2	5742.38	.0001	7998.77	4398.83	.6809
3	5095.50	.0001	7374.38	4217.30	.7687
4	4727.21	.0001	7104.60	4211.60**	.3122
5	4506.55	.0001	7022.72	4298.36	.8318
6	4319.35	.0001	6980.26	4403.35	.8464
7	4141.78	.0001	6946.06*	4510.98	.8601

Table 3-16 (continued)
Communality Estimates

K	Total Estimated Communality (%)	Percentage of Total Estimated Communality Attributed to Each Factor						
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Panel A : based on PCA

		1	2	3	4	5	6	7
1	29.12	100						
2	36.10	80.66	19.34					
3	40.64	71.65	17.18	11.17				
4	43.94	66.27	15.89	10.33	7.51			
5	46.24	62.98	15.10	9.82	7.14	4.95		
6	48.15	60.48	14.50	9.43	6.85	4.76	3.99	
7	49.97	58.27	13.97	9.09	6.60	4.58	3.84	3.64

Panel B : based on MLFA

1	45.33(28.45)	100						
2	66.23(34.84)	79.92	20.08					
3	82.16(39.16)	69.67	19.30	11.03				
4	91.93(42.01)	65.05	18.43	10.45	6.07			
5	98.82(43.79)	63.13	17.58	10.06	5.79	3.45		
6	105.93(45.24)	61.36	16.85	9.73	5.61	3.56	2.90	
7	112.98(46.62)	59.62	16.33	9.53	5.63	3.45	2.85	2.58

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	97.84						
2	98.15	93.01					
3	98.28	94.07	90.06				
4	98.36	94.43	90.58	84.80			
5	98.42	94.56	90.86	85.11	77.33		
6	98.48	94.70	91.16	85.58	79.02	75.41	
7+	98.54	94.86	91.50	86.43	79.60	76.29	74.41

Table 3-17
Summary Information

Number of Securities : 120
Number of Observations : 288
Sample Period : 01/60 - 12/83

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	12465.85	.0001	15103.89	7991.50	.6443
2	10711.64	.0001	13318.21	7316.61	.7468
3	9570.34	.0001	12242.43	6994.83	.8116
4	8929.82	.0001	11749.76	6962.78**	.8443
5	8431.87	.0001	11420.18	7010.44	.8683
6	8118.77	.0001	11305.36	7163.65	.8800
7	7839.65	.0001	11227.15*	7333.34	.8898

Table 3-17 (continued)
Community Estimates

K	Total Estimated Community (%)	Percentage of Total Estimated Community Attributed to Each Factor						
Panel A : based on PCA								
		1	2	3	4	5	6	7
1	37.81	100						
2	43.97	85.99	14.01					
3	48.59	77.81	12.68	9.51				
4	51.81	72.98	11.89	8.92	6.21			
5	54.23	69.72	11.36	8.52	5.92	4.46		
6	56.21	67.27	10.96	8.22	5.71	4.31	3.50	
7	57.98	65.21	10.62	7.97	5.54	4.18	3.40	3.05
Panel B : based on MLFA								
1	57.17(37.13)	100						
2	74.34(42.77)	84.67	15.33					
3	85.99(46.92)	76.72	13.84	9.44				
4	93.19(49.61)	72.93	13.07	9.10	4.90			
5	100.30(51.55)	70.39	12.36	8.64	4.79	3.81		
6	105.07(52.99)	68.64	12.12	8.39	4.64	3.72	2.48	
7	110.12(54.24)	67.07	11.77	8.30	4.63	3.70	2.42	2.12

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	98.28						
2	98.44	91.94					
3	98.51	92.25	89.02				
4	98.55	92.42	89.46	82.03			
5	98.60	92.54	89.66	82.76	86.83		
6	98.63	92.72	89.82	82.98	79.63	72.27	
7+	98.66	92.84	90.14	83.60	80.26	72.68	70.08

Table 3-18
Summary Information

Number of Securities : 120
Number of Observations : 144
Sample Period : 01/60 - 12/71

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	9778.12	.0001	14352.40	7532.58	.6042
2	9043.15	.0001	10632.51	7349.34**	.6873
3	8715.44	.0001	13483.35	7449.97	.7130
4	8415.56	.0001	13367.30	7565.69	.7356
5	8166.28	.0001	13318.25	7713.41	.7514
6	7893.70	.0001	13229.22	7839.66	.7716
7	7666.63	.0001	13201.03*	7994.84	.7855

Table 3-18 (continued)
Communality Estimates

K	Total Estimated Communality (%)	Percentage of Total Estimated Communality Attributed to Each Factor						
Panel A : based on PCA								
		1	2	3	4	5	6	7
1	38.65	100						
2	44.76	86.35	13.65					
3	47.94	80.62	12.75	6.63				
4	50.90	75.93	12.00	6.25	5.82			
5	53.76	71.89	11.37	5.92	5.51	5.32		
6	56.10	68.89	10.89	5.67	5.28	5.10	4.17	
7	58.31	66.28	10.48	5.45	5.08	4.90	4.01	3.79

Panel B : based on MLFA								
1	59.83(37.98)	100						
2	75.39(43.55)	86.07	13.93					
3	83.54(46.16)	81.07	13.33	5.60				
4	91.19(48.59)	77.15	12.57	5.45	4.83			
5	93.72(50.13)	42.03	36.44	11.83	5.01	4.67		
6	118.34(52.50)	70.64	10.81	7.26	4.16	3.76	3.38	
7	108.05(54.23)	38.91	33.85	10.90	4.91	4.42	3.84	3.17

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	98.36						
2	98.48	91.30					
3	98.55	91.76	82.39				
4	98.60	91.97	83.26	81.50			
5+	100.00	97.16	91.73	82.47	81.41		
6	98.82	92.75	89.57	83.10	81.65	80.01	
7+	100.00	97.34	92.17	84.14	82.71	80.60	77.40

Table 3-19
Summary Information

Number of Securities : 120
Number of Observations : 144
Sample Period : 01/72 - 12/83

the Results of Maximum Likelihood Factor Analysis (MLFA)

Number of Factors	Chi-square Value	Probability K Factors Sufficient	Akaike's Information Criterion	Schwarz's Bayesian Criterion	Tucker & Lewis's Reliability Coefficient
1	11178.16	.0001	16338.67	8525.72	.5031
2	10117.87	.0001	15167.32	8116.74	.0089
3	9350.63	.0001	14396.50	7906.55	.6824
4	8860.96	.0001	14011.90	7887.99**	.7238
5	8515.78	.0001	13827.47	7968.02	.7482
6	8247.60	.0001	13748.35	8099.22	.7637
7	8014.31	.0001	13714.52*	8251.59	.7753

Table 3-19 (continued)
Communality Estimates

K	Total Estimated Communality (%)	Percentage of Total Estimated Communality Attributed to Each Factor						
Panel A : based on PCA								
		1	2	3	4	5	6	7
1	38.52	100						
2	46.65	82.57	17.43					
3	52.17	73.84	15.58	10.58				
4	56.37	68.33	14.42	9.79	7.45			
5	59.31	64.95	13.71	9.31	7.08	4.96		
6	61.62	62.51	13.20	8.96	6.82	4.77	3.75	
7	63.87	60.31	12.73	8.64	6.58	4.60	3.62	3.52

Panel B : based on MLFA								
		1	2	3	4	5	6	7
1	60.44(37.85)	100						
2	83.60(45.29)	81.75	18.25					
3	102.12(50.61)	71.58	17.61	10.81				
4	114.20(54.36)	66.67	16.78	10.20	6.35			
5	124.05(56.85)	64.25	15.79	9.75	6.03	4.18		
6	133.09(58.71)	62.48	15.20	9.47	5.79	4.09	2.98	
7	140.92(60.40)	60.77	14.71	9.20	5.95	3.97	2.90	2.49

Squared Canonical Correlations based on MLFA

K	Squared Canonical Correlations for Each Factor with the Variables						
	1	2	3	4	5	6	7
1	98.37						
2	98.56	93.85					
3	98.65	94.73	91.69				
4	98.70	95.04	92.10	87.89			
5	98.76	95.14	92.36	88.21	83.83		
6	98.81	95.29	92.64	88.51	84.49	79.83	
7	98.85	95.40	92.83	89.35	84.84	80.37	77.81

the irrelevant or redundant factors which are unpriced in the market as well as the priced factors.

Second, from the results of SBC which is established to be less inclined to include the redundant factors, the number of factors extracted ranges 2 to 4. This finding is similar to that of RR (1980) in that RR claim only about three or four priced factors are relevant in explaining returns. However, this finding is also preliminary, because SBC itself can not identify whether the factors are priced or not. Also, the number of factors found by SBC is changing over time. This is an interesting issue because if the number of factors are changing over time, the parsimonious aspects of the APT are weakened. Nonetheless, whether the number of priced factors is constant or not should be further evaluated.

Third, when the analysis is performed into the communality estimates using PCA and MLFA, and the squared canonical correlations using MLFA, it is found that both coefficients are the positive function of the sample size, but the negative function of the time observations. Thus, the number of factors found in the return generating process is the positive function of the sample size, but the negative function of the time observations. This finding is inconsistent with that of DFGG (1985). DFGG claim the number of factors are the positive functions of both the sample size and time observation. The difference of the finding is presumably because their conclusion is based on chi-square test which tends to produce exorbitant number of factors.

Whichever the case is , the relevant question is not the number of factors found in the return generating process, but the number of priced or economically relevant factors which affect the economy systematically.

On the Number of Eigenvalues

Chamberlain and Rothschild (CR) (1983) show that, in a limit economy, if the covariance matrix of the asset returns has only k unbounded eigenvalues, then CR's AFS holds even with the weaker condition.

Trzcinka (1986), in his recent application of CR's theoretical extension of the APT, attempts to determine the number of eigenvalues, because in principle it is only necessary to determine the number of unbounded eigenvalues to estimate how many factors are priced. The empirical results of Trzcinka are consistent with the theoretical prediction of Chamberlain (1983) that the one factor model may describe security pricing. Trzcinka's finding is interesting because that is not only inconsistent with the most of previous empirical results of the APT tests, but also that can give a new hindsight that the APT does not necessarily have a multi-factor solution. Of course, if only one factor is relevant in explaining security pricing, that one-factor APT model is, in general, not same with the CAPM, because any efficient portfolio instead of market portfolio can do the role of arbitrage portfolio. Only when each elements of efficient portfolio is exactly same with those of market portfolio, one-factor APT model can be collapsed into the CAPM.

Nonetheless, Trzcinka's conclusion is not that strong. He finds that while only the first eigenvalue dominates the covariance matrix, the first five eigenvalues are growing more distinct. For the values of k larger than one, he asserts that there is no obvious way to choose the number of factors in terms of explained variation. Thus, whether the single factor or multi-factor APT model explains the security pricing better is still unsettled issue.

In this section, Tables 3-20 through 3-22 report the eigenvalues using the sample covariance matrices from PCA and MLFA. Although the first eigenvalue grows large rapidly, when the number of assets increases, simple visual inspection can not tell how many eigenvalues explode without bound.

From these results, it can be summarized that, first, one eigenvalue dominates others at all time periods. However, there is no obvious way to tell how many factors are priced. Second, PCA produces different results from MLFA in terms of eigenvalues. Third, the number of eigenvalues extracted either by PCA or MLFA can be variable instead of constant.

Summary

In this chapter, following the introduction, the approximate factor structure (AFS) of CR (1983) is introduced and compared with the strict factor structure (SFS) of Ross (1976). The third part of this chapter presents some additional characteristics of the AFS and principal component analysis (PCA). The fourth part of this chapter investigates the convergence between the Ross SFS and the CR AFS by employing the orthogonal procrustes rotation

Table 3-20
Eigenvalues Using Sample Covariance Matrices
From the Nested Sequence

n	K							

Panel A : Based on MLFA								

	1	2	3	4	5	6	7	8
30	19.124 (.700)	4.407 (.161)	1.809 (.066)	1.262 (.046)	.807 (.030)	.778 (.029)	.596 (.022)	.581 (.021)
60	44.352 (.580)	10.204 (.134)	4.382 (.057)	2.907 (.029)	2.207 (.023)	1.746 (.023)	1.712 (.022)	1.596 (.021)
90	78.780 (.511)	16.159 (.105)	10.344 (.067)	5.403 (.035)	3.552 (.023)	3.155 (.021)	2.956 (.019)	2.694 (.018)
120	129.368 (.464)	22.734 (.082)	15.386 (.055)	9.218 (.033)	7.650 (.027)	5.562 (.020)	4.713 (.017)	4.560 (.016)

Panel B : Based on PCA								

30	10.236 (.341)	2.483 (.083)	1.592 (.053)	1.283 (.043)	1.007 (.034)	.972 (.032)	.888 (.030)	.872 (.029)
60	19.105 (.318)	4.312 (.072)	2.362 (.039)	1.816 (.030)	1.541 (.026)	1.310 (.022)	1.257 (.021)	1.205 (.020)
90	28.348 (.315)	5.389 (.060)	3.914 (.044)	2.547 (.028)	1.778 (.020)	1.547 (.017)	1.481 (.017)	1.455 (.016)
120	37.814 (.315)	6.158 (.051)	4.622 (.039)	3.214 (.027)	2.424 (.020)	1.975 (.017)	1.769 (.015)	1.691 (.014)

Note: Time period is Jan. 1960 - Dec. 1983 and the numbers in parentheses represent the proportion.

Table 3-21
Eigenvalues Using Sample Covariance Matrices
From the Nested Sequence

n	K							
Panel A : Based on MLFA								
	1	2	3	4	5	6	7	8
30	21.138 (.641)	4.747 (.144)	2.391 (.072)	1.654 (.050)	1.292 (.039)	1.116 (.034)	.885 (.027)	.747 (.023)
60	58.129 (.486)	12.896 (.108)	5.242 (.044)	4.801 (.040)	3.744 (.031)	3.524 (.030)	3.377 (.028)	2.891 (.024)
901	50.732 (.412)	28.784 (.079)	12.885 (.035)	10.753 (.029)	10.102 (.028)	8.998 (.025)	8.263 (.023)	7.443 (.020)
120	485.411 (.375)	79.011 (.061)	41.992 (.033)	35.166 (.027)	31.255 (.024)	29.882 (.023)	26.205 (.020)	23.199 (.018)
Panel B : Based on PCA								
30	10.332 (.344)	2.541 (.085)	1.565 (.052)	1.261 (.042)	1.163 (.039)	1.056 (.035)	.965 (.032)	.906 (.030)
60	19.146 (.319)	4.151 (.069)	2.063 (.034)	1.915 (.032)	1.663 (.028)	1.608 (.027)	1.493 (.025)	1.313 (.022)
90	28.600 (.318)	5.236 (.058)	2.522 (.028)	2.393 (.027)	2.242 (.023)	2.032 (.023)	1.795 (.020)	1.728 (.019)
120	38.655 (.322)	6.109 (.051)	3.176 (.027)	2.958 (.025)	2.858 (.024)	2.340 (.020)	2.210 (.018)	2.119 (.018)

Note: Time period is Jan. 1960 - Dec. 1971 and the numbers in parentheses represent the proportion.

Table 3-22
Eigenvalues Using Sample Covariance Matrices
From the Nested Sequence

n	K							
Panel A : Based on MLFA								
	1	2	3	4	5	6	7	8
30	26.641 (.623)	6.356 (.161)	2.986 (.076)	2.243 (.057)	1.418 (.036)	1.178 (.030)	1.037 (.026)	.843 (.027)
60	73.734 (.482)	20.414 (.134)	9.546 (.062)	6.519 (.043)	5.336 (.035)	3.937 (.026)	3.706 (.024)	3.323 (.022)
90	188.404 (.402)	52.111 (.111)	31.034 (.066)	18.003 (.039)	12.181 (.026)	10.161 (.022)	9.251 (.020)	8.197 (.018)
120	549.807 (.359)	139.42 (.091)	86.609 (.057)	50.551 (.033)	42.548 (.028)	31.105 (.020)	28.084 (.018)	26.689 (.017)
Panel B : Based on PCA								
30	10.542 (.351)	2.734 (.091)	1.903 (.063)	1.428 (.048)	1.139 (.038)	1.013 (.034)	.993 (.033)	.875 (.029)
60	19.731 (.329)	5.163 (.086)	2.888 (.048)	2.241 (.037)	1.944 (.032)	1.575 (.026)	1.452 (.024)	1.327 (.022)
90	29.120 (.324)	6.982 (.078)	4.542 (.051)	3.301 (.037)	2.291 (.026)	1.917 (.021)	1.820 (.020)	1.645 (.018)
120	38.520 (.321)	8.134 (.068)	5.517 (.046)	4.201 (.035)	2.935 (.025)	2.311 (.019)	2.251 (.019)	2.008 (.017)

Note: Time period is Jan. 1972 - Dec. 1983 and the numbers in parentheses represent the proportion.

methodology (OPRM) and well-known varimax rotation methodology (VRM).

To examine the convergence of two factor structures, a necessary and sufficient condition to minimize the differences between the factor loading matrices of SFS and AFS is derived in the context of the OPRM.

Also, the solution of the VRM is mathematically compared with that of the OPRM. Moreover, the test statistics for the average squared difference term between the comparable loadings of MLFA and PCA are presented. The fifth part of this chapter briefly introduces the data and presents the empirical results of convergence. It was found that, by utilizing the OPRM (and by the VRM also), the average squared differences of the comparable loadings of MLFA and PCA are getting smaller when the number of assets grows. The results provide support for the hypothesis of the convergence of SFS and AFS. The results are also consistent with the prediction of the APT in that the variance of idiosyncratic risks should be bounded.

The sixth part of this chapter presents some evidence on the number of factors and eigenvalues. It was found that, first, although one eigenvalue dominates others at all sample periods, there is no obvious way to tell how many factors are priced. Second, the number of factors and/or priced factors can be variable instead of constant. This finding is particularly interesting because if the number of priced factors is changing over time, then it can imply the parsimonious aspects of the APT are lost.

Notes

1. V is positive definite if there exists no riskless assets, and singular if there is a riskless asset with a zero column and a zero row.
2. A square matrix is Gramian if it is symmetric and all of the eigenvalues corresponding to the matrix are positive or equal to zero, i.e.,

$$L(1) \geq L(2) \geq \dots \geq L(N) \geq 0$$

A correlation or covariance matrix is always Gramian.

3. The cosine of the angle q between the vectors x and y is given by

$$\cos q = \frac{x'y}{(x'x)^{1/2}(y'y)^{1/2}}$$

If their inner product vanished such that $x'y = 0$, the vectors lie at right angles to one another. Such vectors are called orthogonal or orthonormal if their length have been normalized to unity. An orthonormal matrix P is a square matrix whose rows are a set of orthonormal vectors. Hence

$$P P' = P'P = I \quad (\text{Morrison 1976, 960})$$

4. The Eckart-Young theorem is as follows:
For any real matrix D , two orthogonal matrices Y and Z can be found for which $Y'DZ$ is a real diagonal matrix with no negative elements. (for proof, see Johnson, 1963)

5. Proof. $TN = \text{tr} (P'B'G)$

$$= \text{tr} [G'B (B'G G'B)^{-1/2} B'G] \quad (\text{from 3-19})$$

$$= \text{tr} [G'B (Y'^{-1} D_y^{-1} Y'^{-1}) B'G] \quad (\text{from 3-18})$$

$$= \text{tr} [B (Y'^{-1} D_y^{-1/2} Y'^{-1}) B'G G']$$

$$= \text{tr} [Y'^{-1} D_y^{-1/2} Y'^{-1} B'G G'B]$$

(permutation leave the trace unchanged)

$$= \text{tr} [D_y^{-1/2} Y'^{-1} B'G G'B Y'^{-1}]$$

$$\begin{aligned}
&= \text{tr} \{ \text{Dy}^{-1/2} \text{Y}^{-1} (\text{Y Dy}^{1/2} \text{Y}') (\text{Y Dy}^{1/2} \text{Y}') \text{Y}'^{-1} \} \\
&\quad (\text{from 3-18}) \\
&= \text{tr} \{ \text{Dy}^{-1/2} \text{Dy}^{1/2} \text{Dy}^{1/2} \} \\
&= \text{tr} \{ \text{Dy}^{1/2} \} \quad \text{Q.E.D.}
\end{aligned}$$

6. What Bartlett (1950) has contrived is an approximate chi-square test of MLFA and PCA which is given by

$$X_{1/2}^2 (n-k)(n-k-1) = -[(T-1) - \frac{1}{6}(2n+5) - \frac{2}{3}k] \log_e X_{n-k}$$

where

$$X_{n-k} = \frac{|R|}{\left[\prod_j r_j \left[(n - \sum_{j=1}^k r_j) / (n-k) \right]^{n-k} \right]}$$

where n is the number of assets

k is the number of factors

T is the number of time observations

r_j is the eigenvalues

$|R|$ is the determinant of correlation matrix

7. The author is grateful to professor Stephen Cosslett for valuable comments of this point.

CHAPTER IV
A MULTI-PERIOD TEST OF K-FACTOR APT

Introduction

Ever since the critique by Roll (1977), the problems associated with the empirical implementations of the capital asset pricing model (CAPM) have led to numerous studies on the arbitrage pricing theory (APT) of Ross (1976). The APT posits a return generating process for the n assets which is a linear function of only k systematic factors, where $k \ll n$.¹ Ross(1976) showed that when portfolios are formed from these assets so that they require no net investment and are riskless, in equilibrium, the portfolios must have an expected return of zero in order to rule out any arbitrage possibilities. An important characteristic of the APT is that unlike the CAPM where a central role is played by the market model, it allows for the existence of many portfolios which are close substitutes.

Numerous tests of the APT have also been performed in the literature. The study by Roll and Ross (RR) (1980) exemplified this literature. Using factor analysis, RR found evidence consistent with the theory and concluded that only three or four priced factors are relevant in explaining expected return. The vast majority of previous tests, though, has not been directly concerned with the dynamic properties of the model. More recently, Dhrymes, Friend, Gultekin, and Gultekin (DFGG) (1985b)

have questioned the appropriateness of the APT in a multi-period context. In particular, they dispute the stability of the number of priced factors over time. However, no formal tests were conducted by DFGG. If the number of factors responsible for the causative, concrete and economic components is large or changing frequently over time, the parsimonious aspects of the APT are weakened. The APT is a useful theory only if the underlying distribution of asset returns allows one to produce diversifiable risk with a representation having $k \ll n$ across assets and over time.

The purpose of this chapter is to formally examine the validity of the APT using time-series data. Specifically, a test of the stability of the k -factor structure over time is constructed and applied to sample data. It has been shown that at any point in time, portfolio returns can be expressed as a linear projection on the historical expected returns and the expected returns of a reference portfolio at the same point in time. This is a convenient expression to use as the null hypothesis of k -factor structure stability can then be examined by testing the null hypotheses of zero intercept and the coefficients that sum to one.

The next part of this chapter provides a brief review of multivariate linear regression approach of Jobson (1982). This is followed by the methodology for testing k -factor stability over time. The fourth part of this chapter provides some implications of the linear combination of the APT that under certain conditions, any minimum variance portfolio can be

expressed as a combination of the arbitrage portfolio obtained from the APT and the minimum variance zero-beta portfolio. The fifth part of this chapter then reports the empirical results of applying the test to a sample data. The sixth part of this chapter presents the results of various tests on economic content of the APT to determine whether there exist additional effects of extraneous variables including the own standard deviation, skewness, and firm size. Also, the related January effect is considered in the context of the APT. This chapter ends with some concluding remarks.

Jobson's Multivariate Linear Regression Approach

By assuming the existence of k linearly independent assets or portfolios, and remaining $(n - k)$ linearly dependent assets or portfolios, Jobson (1982) introduces a multivariate linear regression approach to test the hypothesis of a k -factor APT model.

In particular, the k -factor APT hypothesis is accepted if the intercept term is zero in the multivariate regression function of a vector of $(n - k)$ returns on a vector of k portfolios. In this regard, Jobson presents the return generating process of the APT as follows.

$$\begin{bmatrix} r_{1t} \\ r_{2t} \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} F_t + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} \quad (4-1)$$

where r_{1t} and r_{2t} are the vector of return premium of k linearly independent assets or portfolios and the remaining

(n-k) linearly dependent assets or portfolios respectively;

U_1 and U_2 are the expected value of r_{1t} and r_{1t} ;

B_1 and B_2 are $(k \times k)$ and $((n - k) \times k)$ matrix of factor loadings;

F_t is a $(k \times 1)$ vector of scores on the economically relevant factors;

e_{1t} and e_{2t} are $(k \times 1)$ and $((n - k) \times 1)$ vector of error terms respectively.

Subtraction of the product of $B_2 B_1$ and the first equation of (4-1) from the second equation of (4-1) leads to

$$r_{2t} = (U_2 - B_2 B_1^{-1} U_1) + (B_2 B_1^{-1} r_{1t} + e_{2t} - B_2 B_1^{-1} e_{1t}) \quad (4-2)$$

Also from the conclusion of the APT, the expected value of return premium vector can be expressed

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} RP_t \quad (4-3)$$

where RP_t is a $(k \times 1)$ vector of risk premia.

Thus, it is straightforward to show from the equation (4-3) that

$$\begin{aligned} U_2 &= B_2 RP_t \\ &= B_2 B_1^{-1} U_1 \end{aligned} \quad (4-4)$$

Hence, the first term in equation (4-2) is zero. To interpret the equation (4-2) as the multivariate linear regression

approach, Jobson needs a strong restriction on the error term of equation (4-2). Specifically, assuming the return premium vector r_{1t} to represent the return premiums of a set of k linearly independent assets or portfolios obtained from a large set of assets, Jobson claims that the error term e_{1t} could be ignored.

However, it seems that there exist several weaknesses in Jobson's analysis. First, it is difficult to assume the existence of k linearly independent assets or portfolios. The factors in a k -factor APT model refer some linear combination of economically relevant or systematic portion of each asset. Thus, the assumption of k linearly independent assets or portfolios is not only unnecessary for the test of a k -factor APT model, but also overly strong.

Second, Jobson utilizes the return premium vector r_t and mean premium vector U_t instead of observed return vector R_t and mean vector E_t . However, in the empirical sense, it is not easy to estimate the risk-free or zero-beta rate because of the identification problem of the risk-free and/or zero-beta assets.

Finally, Jobson's restriction on the error term is unreasonable because the noise term will increase rather than decrease if assets become large when the number of systematic factors, k , is fixed.

The following methodology eliminates these problems. To be specific, by using the concept of reference or base portfolio, we do not need the strong assumption of the existence of k linearly independent assets or portfolios. Furthermore, observed return vector R_t and mean vector E_t instead of the return premium vector

and mean premium vector are utilized. Hence, there is no identification problem of the risk-free or zero-beta assets which are unobservable. In addition, it is shown that making a strong restriction on error term to test a K-factor APT is not necessary.

Also, our methodology can be extended to a multi-period context, whereas Jobson's analysis is constrained to a single period.

Methodology

Tests for the stability of K-factor structure over time.

The return generating process of the APT can be written as

$$R_t = E_t + B F_t + e_t \quad t=1,2,\dots, T \quad (4-5)$$

where R_t is an $(n \times 1)$ vector of observed random returns

E_t is an $(n \times 1)$ vector of expected returns

B is an $(n \times K)$ matrix of factor loadings, assumed to be of rank K

F_t is a $(K \times 1)$ vector of scores on the systematic factors

e_t is an $(n \times 1)$ vector of idiosyncratic noise terms

The unobserved systematic factor scores are assumed to have zero means, and to be linearly independent. The error terms are assumed to be uncorrelated with F_t , uncorrelated with each other, and assumed to have zero mean².

Two-period Case

It is easiest to construct a test of the null hypothesis in which the number of economically relevant (or 'priced') k factors is constant over time in the context of two periods, t_1 and t_2 . Generalizations to multi-periods should be obvious. The return generating processes for two portfolios in the two periods can be expressed as the following mutually exclusive equation system:

$$\begin{bmatrix} R_b \\ R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} E_b \\ E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} B_b \\ B_1 \\ B_2 \end{bmatrix} F_t + \begin{bmatrix} e_b \\ e_1 \\ e_t \end{bmatrix} \quad (4-6)$$

where R_b is an $(n \times 1)$ vector of observed returns on a base portfolio observed at time t_1 (In this paper, a "base" portfolio is interchangeably used with a "reference" portfolio).

R_1 is an $(n \times 1)$ vector of returns on a comparison portfolio observed at time t_1 .

R_2 is an $(N \times 1)$ vector of returns on the same comparison portfolio as R_1 but observed at time t_2 .

The notation of the other variables conforms to that of R_t .

Subtracting R_b from R_1 and R_2 respectively, (4-6) can be rewritten as

$$\begin{bmatrix} R_1^o \\ R_2^o \end{bmatrix} = \begin{bmatrix} E_1^o \\ E_2^o \end{bmatrix} + \begin{bmatrix} B_1^o \\ B_2^o \end{bmatrix} F_t + \begin{bmatrix} e_1^o \\ e_2^o \end{bmatrix} \quad (4-7)$$

where $R_1^O = R_1 - R_b$, $E_1^O = E_1 - E_b$, $B_1^O = B_1 - B_b$

$R_2^O = R_2 - R_b$, $E_2^O = E_2 - E_b$, and $B_2^O = B_2 - B_b$

Solving for F_t using the first equation of (4-7) results in

$$F_t = (B_1^{O'} B_1^O)^{-1} B_1^{O'} (R_1^O - E_1^O - e_1^O) \quad (4-8)$$

Substitution of (4-8) into the second equation of (4-7) leads to

$$\begin{aligned} R_2^O &= [E_2^O - B_2^O (B_1^{O'} B_1^O)^{-1} B_1^{O'} E_1^O] + B_2^O (B_1^{O'} B_1^O)^{-1} B_1^{O'} R_1^O \\ &\quad + (e_2^O - B_2^O (B_1^{O'} B_1^O)^{-1} B_1^{O'} e_1^O) \end{aligned} \quad (4-9)$$

Equation (4-9) can be further simplified by invoking the APT. If the central conclusion of the APT is true, the risk-free rate or zero-beta rate, RP_0 , and risk premiums, RP_k , are time invariant, and the expected portfolio returns can be expressed as

$$\begin{bmatrix} E_b \\ E_1 \\ E_2 \end{bmatrix} = RP_0 + \begin{bmatrix} B_b \\ B_1 \\ B_2 \end{bmatrix} RP_k \quad (4-10)$$

Subtracting the first equation from the second and third equations, we obtain

$$\begin{bmatrix} E_1^O \\ E_2^O \end{bmatrix} = \begin{bmatrix} B_1^O \\ B_2^O \end{bmatrix} RP_k \quad (4-11)$$

where $E_1^O = E_1 - E_b$, $E_2^O = E_2 - E_b$, $B_1^O = B_1 - B_b$, and

$$B_2^O = B_2 - B_b$$

Therefore, under the null hypothesis that RP_0 and RP_k are stable over time, the following relationship is easily seen from equation (4-11):

$$E_2^O = B_2^O (B_1^O, B_1^O)^{-1} B_1^O, E_1^O \quad (4-12)$$

From equation (4-6), (4-9), and (4-12), the realized returns on the comparison portfolio at t_2 can be written as the multivariate linear regression form

$$\begin{aligned} R_2 &= (E_2^O - X E_1^O) + X (R_1 - e_1) + (I - X)(R_b - e_b) + e_2 \\ &= (E_2^O - X E_1^O) + X (E_1 + B_1 F_t) + (I - X)(E_b + B_b F_t) \\ &\quad + e_2 \end{aligned} \quad (4-13)$$

where $X = B_2^O (B_1^O, B_1^O)^{-1} B_1^O$ and

I is an $(N \times N)$ identity matrix.

If we do not allow any additional restrictions on e_1 and e_b , the estimation of intercept term and coefficients in equation (4-13) would require an errors-in-variables approach because both explanatory variables, $(E_1 + B_1 F_t)$ and $(E_b + B_b F_t)$, are unobservable. Both terms, however, can be estimated by employing maximum likelihood factor analysis (MLFA) and/or principal component analysis (PCA). Whereas MLFA yields more efficient estimators for small samples, estimation of F_t is not easy and estimators suffer the well-known nonuniqueness problem.³ On the

other hand, PCA provides unique estimators, but ones which are less efficient than those of MLFA for small samples. For large enough samples, the two approaches yield asymptotically equivalent estimators (CR) (1983). In this study we adopt both statistical techniques.

It can also be shown that measurement errors in using estimated rather than observed regressors are not serious (Appendix B).⁴ Tests of the stability of a k-factor APT model over time can then be performed by examining a zero intercept term and the coefficients in the multivariate linear regression of R_2 on the estimated value of $(E_1 + B_1 F_t)$ and $(E_1 + B_b F_t)$. Generalizations to the multi-period case are illustrated in the following section.

Multi-Period Case

Generalizations of the multivariate linear regression with the errors-in-variable (MLR-EV) approach to multi-periods should be obvious. To illustrate this, consider the return generating processes for two portfolios in the multi-periods.

$$\begin{bmatrix} R_b \\ R_1 \\ \vdots \\ R_p \end{bmatrix} = \begin{bmatrix} E_b \\ E_1 \\ \vdots \\ E_p \end{bmatrix} + \begin{bmatrix} B_b \\ B_1 \\ \vdots \\ B_p \end{bmatrix} F_t + \begin{bmatrix} e_b \\ e_1 \\ \vdots \\ e_p \end{bmatrix} \quad (4-6-1)$$

where R_p is an $(N \times 1)$ vector of returns on the comparison portfolio as R_1 but observed at time t_p , and other terms are defined as before.

Subtracting R_b from R_1 through R_p respectively as in two-period case, (4-6-a) can be expressed as

$$\begin{bmatrix} R_1^o \\ : \\ R_p^o \end{bmatrix} = \begin{bmatrix} E_1^o \\ : \\ E_p^o \end{bmatrix} + \begin{bmatrix} B_1^o \\ : \\ B_p^o \end{bmatrix} F_t + \begin{bmatrix} e_1^o \\ : \\ e_p^o \end{bmatrix} \quad (4-7-a)$$

where $R_p^o = R_p - R_o$, $E_p^o = E_p - E_b$, $B_p^o = B_p - B_o$, and

$$e_p^o = e_p - e_b$$

Subtraction of (4-8) into the last equation of (4-7-a) yields

$$R_p^o = (E_p^o - X_p E_1^o) + X_p R_1^o + (e_p^o - X_p e_1^o) \quad (4-9-a)$$

$$\text{where } X_p = B_p^o (B_1^o, B_1^o)^{-1} B_1^o,$$

Rearranging (4-9-a) in terms of R_p using the APT equation, we can obtain

$$\begin{aligned} R_p &= (E_p^o - X_p E_1^o) + X_p (E_1 + B_1 F_t) + (I - X_p)(E_b + B_b F_t) \\ &\quad + e_p \end{aligned} \quad (4-13-a)$$

This is because the multi-period case boils down to the simple two-period case. To be specific, consider the APT equation when the risk-free rate or zero-beta rate, RP_o , and risk premiums, RP_k , are time invariant.

$$\begin{bmatrix} E_b \\ E_1 \\ \vdots \\ E_p \end{bmatrix} = RP_0 + \begin{bmatrix} B_b \\ B_1 \\ \vdots \\ B_p \end{bmatrix} RP_k \quad (4-10-a)$$

Subtracting the first equation from the second up to the last equation respectively, we can obtain

$$\begin{bmatrix} E_1^0 \\ \vdots \\ E_p^0 \end{bmatrix} = \begin{bmatrix} B_1^0 \\ \vdots \\ B_p^0 \end{bmatrix} RP_k \quad (4-11-a)$$

Expressing E_p in terms of E_1 , equation (4-11-a) can be written as

$$\begin{aligned} E_p^0 &= B_p^0 RP_k \\ &= B_p^0 (B_{p-1}^{0'} B_{p-1}^0)^{-1} B_{p-1}^{0'} E_{p-1}^0 \\ &\quad \vdots \\ &= B_p^0 (B_2^{0'} B_2^0)^{-1} B_2^{0'} E_2^0 \\ &= B_p^0 (B_2^{0'} B_2^0)^{-1} B_2^{0'} B_2^0 (B_1^{0'} B_1^0)^{-1} B_1^{0'} E_1^0 \\ &= B_p^0 (B_1^{0'} B_1^0)^{-1} B_1^{0'} E_1^0 = X_p E_1^0 \quad (4-12-a) \end{aligned}$$

From equations (4-12-a) and (4-13-a), the first term in equation (4-13-a) is zero and the sum of coefficients of explanatory variables is identity matrix.

In addition, it can be shown that the generalized Jobson's approach is a special case of this multivariate linear regression with the errors-in-variable approach. To illustrate this, define

$R_t^i(n)$ as the returns on i -th sample which has n securities at time t such that

$R_{t_1}^b(n)$ is an $(n \times 1)$ vector of returns on a reference portfolio at time t_1 ,

$R_{t_1}^j(n)$ is an $(n \times 1)$ vector of returns on a j -th comparison portfolio at time t_1 ,

$R_{t_2}^j(n)$ is an $(n \times 1)$ vector of returns on a j -th comparison portfolio at time t_2 , and

$R_{t_1}^i(n)$ is an $(n \times 1)$ vector of returns on the i -th comparison portfolio at time t_1 respectively.

Then the equation (4-13) can be extended to Jobson's two subsets and one period approach.

$$\begin{aligned} R_t^i(n) &= [E_t^{io}(n) - X_t E_t^{jo}(n)] + X_t [R_t^j(n) - e_t^j(n)] \\ &\quad + (I - X_t)[R_t^b(n) - e_t^b(n)] + e_t^i(n) \quad (4-13-b) \end{aligned}$$

$$\text{where } E_t^{io}(n) = E_t^i(n) - E_t^b(n)$$

$$E_t^{jo}(n) = E_t^j(n) - E_t^b(n)$$

$$X_t = B_t^o (B_j^o, B_j^o)^{-1} B_j^o, \text{ at time } t, \text{ and}$$

the notation of the other variables conforms to that of $R_t^i(n)$.

The equation (4-13-b) can be called as generalized Jobson's case. Accordingly, equation (4-13) for the multi-period case can be rewritten as

$$R_{tp}^j(n) = [E_{tp}^{jo}(n) - X_p E_{t_1}^{jo}(n)] + X_p [R_{t_1}^j(n) - e_{t_1}^j(n)]$$

$$+ (I - X_p)[R_{t1}^b(n) - e_{t1}^b(n)] + e_{tp}^j(n) \quad (4-13-c)$$

$$\text{where } X_p = B_{tp}^{jo} (B_{t1}^{jo}, B_{t1}^{jo})^{-1} B_{t1}^{jo}$$

As long as the number of sample size is n , equations (4-13-b) and (4-13-c) can be shown as a special case of the generalized case.

$$\begin{aligned} R_{tp}^i(n) &= [E_{tp}^{io}(n) - X_p^i E_{t1}^{jo}(n)] + X_p^i [R_{t1}^j(n) - e_{t1}^j(n)] \\ &+ (I - X_p^i)[R_{t1}^b(n) - e_{t1}^b(n)] + e_{tp}^j(n) \quad (4-13-d) \end{aligned}$$

$$\text{where } X_p^i = B_{tp}^{io} (B_{t1}^{jo}, B_{t1}^{jo})^{-1} B_{t1}^{jo}$$

Therefore, tests of the stability of multi-period K-factor APT model can be performed by examining a zero intercept term and the coefficients in the multivariate linear regression of R_p on the estimated value of $(E_1 + B_1 F_t)$ and $(E_b + B_b F_t)$

More specifically, the null hypothesis is

$$H_0 : E_p^0 - X_p E_1^0 = 0 \text{ and } X_p + (I - X_p) = I$$

This can be examined by using the standard F-test. An alternative approach is to use the likelihood ratio test.

Likelihood Ratio Tests.

Equation (4-13) can be simply written in cross-sectional multiple regression form as⁵

$$R_2 = b_0 + b_1 R_1^* + b_2 R_b^* + e_2 \quad (4-14)$$

where $b_0 = E_2 - X E_1 = 0$, $b_1 = S_{ii}$, $b_2 = 1 - X_{ii}$

R_1^* = estimated value of $(E_1 + B_1 F_t)$, and

R_2^* = estimated value of $(E_b + B_b F_t)$

Likelihood ratio tests can be used to test

$H_0 : b_0 = 0 \text{ and } b_1 + b_2 = 1$

$H_A : b_0 \neq 0 \text{ and/or } b_1 + b_2 \neq 1$

For T independent samples from the multivariate normal

$R_t \sim N(E, \Sigma)$ where $R_t (n \times 1)$, $E(n \times 1)$, $\Sigma(n \times n)$,

$t = 1, 2, \dots, T$, the likelihood ratio is given by

max $L(b_i)$ over values of b_i specified by H_0

$$\begin{aligned}
 L &= \frac{\max L(b_i) \text{ over all values of } b_i}{\max L(b_i) \text{ over values of } b_i \text{ specified by } H_0} \\
 &= \frac{\sup L_r(b_i; R_1, \dots, R_n) |S_r|}{\sup L_u(b_i; R_1, \dots, R_n) |S_u|} \quad i = 0, 1, 2 \quad (4-15)
 \end{aligned}$$

where $L(b_i)$ is the likelihood function of b_i , and

$|S_r|$ and $|S_u|$ are the determinants of covariance matrix estimated from the residuals of the restricted and unrestricted model respectively.

The logarithm of the likelihood function is given by

$$\log L = - \frac{NT}{2} \log 2\pi - \frac{T}{2} \log |\Sigma|$$

$$- \frac{1}{2} \Sigma_t (R_t - E)' \Sigma^{-1} (R_t - E) \quad (4-16)$$

The unrestricted maximum likelihood estimators are given by

$$\bar{R} = \frac{1}{T} \Sigma_t R_t \quad \text{and}$$

$$S = \frac{1}{T} \Sigma_t (R_t - \bar{R})(R_t - \bar{R})' \quad (4-17)$$

Using the fact that $-2 \log L$ is asymptotically a Chi-square distribution, we have

$$-2 \log L = T (\log |S_x| - \log |S_u|) \sim \chi^2 \quad (4-18)$$

Tests for the stability of zero-beta rate.

In our test above, use is made of the fact that RP_0 is time invariant under the APT. Rejection of the null hypothesis of k-factor APT stability may therefore be due to the failure of the risk-free or zero-beta rate to be constant over time. Hence, we consider a test designed specifically for RP_0 .

It is also of interest to note that previously, DFEG (1985a, 1985b) found that tests for the equality of intercepts to the

risk-free rate were accepted for groups of 30 and 60 stocks at 5 percent level and rejected for groups of 90 stocks at 10 percent level. They also conducted tests to determine if all intercepts are equal such that

$$H_0 : RP_0(I, t) = RP_0(J, t) = RP_0 \quad t = 1, 2, \dots, T,$$

where I and J are different subsets of assets.

However, they were unable to reject H_0 .

This study is concerned with the multi-period behavior of risk-free rate or zero-beta rate so that the null hypothesis of interest is

$$H_0 : RP_0(I, t_i) = RP_0(I, t_j) = RP_0 \quad i \neq j$$

This can be examined by using RR (1980) two-step factor analytic approach and employing Chow test, i.e.,

$$F = \frac{(RRSS - URSS) / 1}{URSS / 2(N - k - 1)}$$

where RRSS and URSS are the restricted and unrestricted residual sum of squares respectively, and k is the number of explanatory variables.

Implications

The implication of the linear combination of the APT is that with unrestricted short-selling in equilibrium, any minimum variance portfolio can be expressed as a combination of the efficient index portfolio, which we call arbitrage portfolio I,

obtained from the APT model and the minimum variance zero-beta portfolio, Z , whose return is orthogonal to the return on I , R_I . The first step is to show that the set of minimum variance portfolios can be generated as combinations of any two minimum variance portfolios. The second step is to show that a minimum variance zero-beta portfolio always exists.

To show the first step, consider the APT equation (1-2)

$$E_i = RP_0 + b_{i1} RP_1 + \dots + b_{ik} RP_k$$

Now, consider a portfolio, Z , which is zero-beta on all factors but the p -th then,

$$Z' b_j = 0 \text{ for } j \neq p$$

$$Z' b_j = 1 \text{ and } Z' l_j = 1 \text{ for } j=p \quad (4-19)$$

where l is an $(n \times 1)$ unit vector.

Using (4-19), the risk premium on the p -th portfolio can be expressed as

$$E^p - RP_0 = Z' (E_I - RP_0) = RP_p \quad (4-20)$$

where E_I is the ex-ante return on any efficient index portfolios with minimum idiosyncratic risk in the APT model.

Likewise, if the APT holds, the equilibrium return on each RP_j is given by the APT as

$$\begin{bmatrix} RP_1 \\ RP_2 \\ \vdots \\ RP_k \end{bmatrix} = \begin{bmatrix} Z_{RP1} \\ Z_{RP2} \\ \vdots \\ Z_{RPk} \end{bmatrix} (E_I - RP_o) \quad (4-21)$$

Substituting (4-21) into (1-2) yields

$$\begin{aligned} E_i &= RP_o + b_{i1} RP_1 + \dots + b_{ik} RP_k \\ &= RP_o + b_{i1} [Z_{RP1} (E_I - RP_o)] + \dots \\ &\quad + b_{ik} [Z_{RPk} (E_I - RP_o)] \\ &= RP_o + (b_{i1} Z_{RP1} + \dots + b_{ik} Z_{RPk}) (E_I - RP_o) \\ &= RP_o + B_i (E_I - RP_o) \end{aligned} \quad (4-22)$$

If the CAPM holds for all securities, as well as for portfolios of securities, then equation (4-22) can be written as

$$E_i = RP_o + B_i (E(R_m) - RP_o) \quad (4-22-a)$$

In the case of $E_I = E(R_m)$, the APT model collapses into the CAPM. However, the market portfolio needs not play a special role in the APT, since one can approximately duplicate the factors with linear combinations of the assets.

Roll (1977) also points out that there is nothing unique about the market portfolio. Roll further claims that it is always possible to choose any efficient portfolio as an index and to find the minimum variance portfolio which is orthogonal to the

selected efficient index. Therefore, as in Black (1972), the returns on different minimum variance portfolios can be obtained by varying x :

$$R_e = x R_Z + (1 - x) R_I \quad (4-23)$$

When minimum variance portfolios are formed in this fashion, the expected return and the variance on a minimum variance portfolio are

$$E(R_e) = x E(r_Z) + (1 - x) E(R_I) \quad (4-23-a)$$

$$\text{Var}(R_e) = x^2 \text{Var}(R_Z) + (1 - x)^2 \text{Var}(R_I) \quad (4-23-b)$$

since R_Z and R_I are orthogonal.

The second step is straightforward since the range of expected returns of minimum variance portfolios is not bounded, with any given level of expected return, there should exist a minimum variance portfolio. The only problem associated with this interpretation is whether the multi-beta term in equation (4-22), $(b_{i1} Z_{RPI} + \dots b_{ik} Z_{RPK})$, can be represented by a single measure of systematic risk, B_i .

This is the debate between Dybvig and Ross (1985) and Shanken (1985). Even when the multi-beta term can be transformed into a single measure, B_i , it is no longer equal to that of the CAPM, as E_I can be any efficient portfolio with minimum idiosyncratic risks. Such an efficient portfolio is called as "well-diversified" portfolio. Whether the multi-beta term can be transformed into a single term as in the CAPM should be further evaluated and beyond the scope of this study.

The implication for the empiricists is that, as GT (1985a) has stated, the portfolios with minimum idiosyncratic risk in a factor model are observable, while the market portfolios are unobservable. GT further indicate that MLFA asymptotically choose a set of proxy portfolios that are locally efficient if and only if the APT holds.⁶

Empirical Results

Data

The data for this study are obtained from 1984 CRSP monthly tape. The monthly returns of 60 and 100 randomly selected securities listed on the NYSE for the 1960 - 1983 time period are used. The data are briefly described in Table 4-1. Since maximum likelihood factor analysis relies on test statistics that assume multivariate normality, monthly data are utilized as they have been shown to be more reliable than daily return data by Fama (1976).

In the process of constructing base portfolio and comparison portfolio, no more than four securities in each four digit SIC code among 32 industries randomly selected from Appendix B of 1984 COMPUSTAT manual are included in both portfolios. Moreover, in an effort to make both portfolios comparable, SIC code of each component of 32 randomly chosen industries is kept the same in both portfolios and there is no overlapping of securities between portfolios. The base portfolio and the comparison portfolio constructed in this manner are used for multivariate linear regression tests on the stability of k-factor APT model over

time. For the stationarity test of risk-free or zero-beta rate, only the base portfolio is used.

Results of Stability Test of Multi-period K-factor APT Model

To see how well the data support the models, we examine the result of cross-sectional regression of observed returns on comparison portfolio at t_2 on estimated values of returns on comparison portfolio at t_1 and those of base portfolio at t_1 in each of the subperiods. The independent variables, R_1 and R_2 , are estimated by employing PCA. MLFA is also employed only for comparison purpose with the results of PCA. It will be useful to reiterate the equation (4-14) here.

$$R_2 = b_0 + b_1 R_1^* + b_2 R_b^* + e_2$$

To examine the stability of k-factor APT model over time, the following three tests are employed.

$$\text{Test 1 : } H_0 : b_0 = 0$$

$$\text{Test 2 : } H_0 : b_1 + b_2 = 1$$

$$\text{Test 3 : } H_0 : b_0 = 0 \text{ and } b_1 + b_2 = 1$$

Test 1 investigates whether the intercept term of equation (4-14) is zero, test 2 examines whether the coefficients of independent variables sum to one, and test 3 is the joint test of both test 1 and test 2. The empirical results from cross-sectional regressions of equation (4-14) are presented in Tables 4-2 through 4-4.

From Table 4-2, it can be seen that joint tests are rejected over 61 percent for subperiods I and II and over 70 percent for subperiods III and IV at 5 percent significance level. Thus, we can conclude that the stability of 5-factor structure is violated for these time periods. These preliminary findings are disconcerting because if K-factor structures are unstable over time, the parsimonious aspects of APT are lost and the practical usefulness of multi-period APT is limited.

To investigate the stability of K-factor structure more thoroughly, employing PCA for the case of $K=1$ up to 5 when the number of assets is 60 and 100 respectively, detailed results are presented in Tables 4-3 through 4-4. Table 4-3 illustrates the similar results with those of Table 4-2. In particular, joint tests are rejected over 70 percent for subperiods I and II, and for subperiods III and IV except the case of $K=4$ at 5 percent significance level. However, while the results of MLFA do not provide any distinct pattern for longer subperiods, the results of PCA tend to reduce the percent rejection rates for longer subperiods. These phenomena are most distinct in test 1, but not very prevalent for test 3.

Even when the number of assets increase up to 100, the previous results do not seem to be changed seriously. Table 4-4 shows the results. The percentage rejection rates of test 3 slightly increased over 77 percent for all the cases of $K=1$ up to 5 of subperiod III and IV at 5 percent significance level. While the percentage rejection rates of test 2 slightly reduced for these time periods, those of test 1 relatively increased, leading to the overall increase of percentage rejection rates of test 3.

In addition, it is not possible to distinguish any systematic pattern of K-factor structure based upon these results. In other words, whatever the number of k is, the percentage rejection rates of test 3 are over 70 percent for subperiods I and II, and over 70 and 77 percent for subperiods III and IV when the number of assets is 60 and 100 respectively. Therefore, as before, it can be concluded that the stability of k -factor structure is violated for these time periods whatever the number of factors is, implying that the parsimonious aspects of APT are lost and the practical usefulness of multi-period APT is limited.

Of course, this is not to say that the theory is rejected. However it is argued in this paper that practical applicability of the APT for investment decision purpose should be viewed as "questionable" based on this empirical evidence.

Results of Tests of Risk-free Rate Stability over Time

An important implication of the APT is that the constant term in equation (4-10) corresponds to the risk-free rate or at least the return on a zero-beta asset. In conjunction with this implication, RR two-step factor analytic approach requires time-invariant risk-free rate. However it is doubtful that risk-free rate stays stable over time. If the risk-free rate is stochastic, the results of the regression tests may be affected. To investigate this possibility of nonstationary risk-free rate, we use RR two-step factor analytic approach. The relevant results are summarized in Tables 4-5 and 4-6.

Both estimation procedures, GLS and OLS are reported so that comparisons with previous studies may be made. For example, RR used GLS while Chen (1983) used OLS. Both GLS and OLS estimations yield similar results, with GLS exhibiting a stronger statistical significance of the intercept term and a higher R^2 . Both the GLS and OLS results indicate that the risk-free rate for short subperiods, I and II, is by no means nonstationary. On the other hand, the stationarity of risk-free rate for longer time periods III and IV is obviously violated. It is seen in Table 4-6.

Panel B in Table 4-5 presents tests for the equality of intercepts to the risk-free rate. Intercept terms for short subperiods, I and II, are not significantly different from .5 percent per month which was the number used by RR. However, intercept terms for longer time periods, III and IV, are significantly different from .5 percent per month. It is seen in Table 4-6.

Hence, the empirical finding of this section is that the stability test of risk-free rate as well as tests for the equality of intercepts to the risk-free rate are supported only for a time period of 6 years, but are rejected for a longer period of 12 years.

This finding is similar to the CAPM study of Stambaugh (1981) in which he reports the stationarity assumption holds well over the 6-year period used in his study. This result is also consistent with Fama and MacBeth (1973) study in that they assume the parameters of the distribution remain constant over 5 to 8

Table 4-1
Data Description

Source	: Center for Research on Security Prices Graduate School of Business University of Chicago 1984 Monthly RReturns File
Selection Criterion	: Randomly selected 60 and 100 individual securities from those listed on the NYSE on both January 1960 and December 1983.
Basic Data Unit	: Return adjusted for all capital changes and including dividends.
Sample Time Period	: I. January 1972 - December 1977 II. January 1978 - December 1983 III. January 1960 - December 1971 IV. January 1972 - December 1983 V. January 1960 - December 1983

Note: Sample subperiod I and II include 72 months, subperiod III and IV represent 144 months, and time period V has 288 months respectively.

Table 4-2
 Percentage Rejection Rates for Three Tests
 Using Cross-Sectional Regressions (n=60)

Time Period	Test 1			Test 2			Test 3		
	a=.05	a=.10	1=.15	a=.05	a=.10	a=.15	a=.05	a=.10	a=.15
I & II	29.17 (21)	40.28 (29)	45.83 (33)	6.94 (5)	11.11 (8)	16.67 (12)	61.11 (44)	62.5 (45)	65.28 (47)
III & IV	19.44 (28)	25.69 (37)	29.86 (43)	7.64 (11)	14.58 (21)	20.83 (30)	70.14 (101)	75.0 (108)	79.17 (114)

Note: The numbers in parentheses represent the number of significant months. Note that subperiod I and II include 72 months and subperiod III and IV represent 144 months respectively. The a values denote significance levels. Independent variables are estimated by utilizing MLFA

when $K=5^7$.

Table 4-3
 Percentage Rejection Rates for Three Tests
 Using Cross-Sectional Regressions (n=60)

Time Period	Test 1			Test 2			Test 3		
	a=.01	a=.05	1=.10	a=.01	a=.05	a=.10	a=.01	a=.05	a=.10
Panel A : Results on K=1									
I & II	34.72 (25)	44.44 (32)	51.39 (37)	4.17 (3)	13.89 (10)	22.22 (16)	63.89 (46)	72.22 (52)	77.78 (56)
III & IV	15.28 (22)	29.17 (42)	38.19 (55)	0 (0)	6.25 (9)	15.28 (22)	55.56 (80)	70.83 (102)	75.0 (108)
Panel B : Results on K=2									
I & II	30.56 (22)	41.67 (30)	52.78 (38)	2.78 (2)	12.5 (9)	20.83 (15)	62.5 (45)	76.39 (55)	79.17 (57)
III & IV	13.19 (19)	25.69 (37)	29.86 (43)	.69 (1)	10.42 (15)	16.67 (24)	58.33 (84)	72.92 (105)	79.86 (115)
Panel C : Results on K=3									
I & II	31.94 (23)	44.44 (32)	52.78 (38)	5.56 (4)	12.5 (9)	25.0 (18)	61.11 (44)	76.39 (55)	80.56 (58)
III & IV	10.42 (15)	22.92 (33)	32.64 (47)	0 (0)	11.11 (16)	18.75 (27)	59.03 (85)	72.22 (104)	79.17 (114)
Panel D : Results on K=4									
I & II	29.17 (21)	47.22 (34)	51.39 (37)	5.56 (4)	13.89 (10)	26.39 (19)	61.11 (44)	76.39 (55)	79.17 (57)
III & IV	6.94 (10)	22.92 (33)	27.78 (40)	0 (0)	8.33 (12)	16.67 (24)	56.94 (82)	67.36 (97)	77.78 (112)
Panel E : Results on K=5									
I & II	33.33 (24)	44.44 (32)	51.39 (37)	4.17 (3)	15.28 (11)	19.44 (14)	62.5 (45)	75.0 (54)	77.78 (56)
III & IV	12.5 (18)	20.14 (29)	32.64 (47)	1.39 (2)	8.33 (10)	15.97 (23)	58.33 (84)	73.61 (106)	77.08 (111)

Note: the numbers in parentheses represent the number of significant months. Note that subperiods I and II include 72 months and subperiod III and IV represent 144 months respectively. The a values denote significance levels. Independent variables are estimated by employing PCA.

Table 4-4
Percentage Rejection Rates for Three Tests
Using Cross-Sectional Regressions (n=100)

Time	Test 1			Test 2			Test 3		
Period	a=.01	a=.05	1=.10	a=.01	a=.05	a=.10	a=.01	a=.05	a=.10
Panel A : Results on K=1									
III &	18.75	29.86	46.97	.69	7.64	14.58	69.44	78.47	79.86
IV	(27)	(43)	(59)	(1)	(11)	(21)	(100)	(113)	(115)
Panel B : Results on K=2									
III &	14.58	27.08	36.81	.69	9.03	16.67	70.14	81.25	82.64
IV	(21)	(39)	(53)	(1)	(13)	(24)	(101)	(117)	(119)
Panel C : Results on K=3									
III &	16.67	27.08	36.81	2.08	7.64	14.58	71.53	78.47	82.64
IV	(24)	(39)	(57)	(3)	(11)	(21)	(103)	(113)	(119)
Panel D : Results on K=4									
III &	17.34	27.78	36.81	.69	9.03	15.97	69.44	77.08	81.94
IV	(25)	(40)	(53)	(1)	(13)	(23)	(100)	(111)	(118)
Panel E : Results on K=5									
III &	18.06	29.17	39.58	2.78	9.03	14.58	68.75	77.08	81.94
IV	(26)	(42)	(57)	(4)	(13)	(21)	(99)	(111)	(118)

Note: The numbers in parentheses represent the number of significant months respectively. The a value denotes significance levels. Independent variables are estimated by employing PCA. Since the number of time observations should be larger than the number of assets for analysis, only subperiods III & IV are used.

Table 4-5
Stationarity Test of Risk-free Rate
from RR Second-step Regression of Arithmetic
Mean Sample Returns on Factor Loadings.

Time Period	Results from GLS	Results from OLS

Panel A :Testing stationarity of risk-free rate		

I vs. II	.9175	.8602
II vs. I	.9245	.8686
III vs. IV	.0971*	.1495
IV vs. III	.0052**	.0167*

Panel B : Testing $RP_0 = .005$ per month (6% annum equivalent)		

I	.1057 (.2298)	.1482 (.2172)
II	.1120 (.1740)	.1286 (.1722)
III	.0019**(.2030)	.0037**(.1899)
IV	.0089**(.2667)	.0106**(.2666)

Note: the numbers represent the probability of F greater than the critical value when Chow test is employed. The numbers in parentheses (in Panel B) are R^2 from RR second-step regression.

* indicates rejection of the null hypothesis at the 10% level.

** indicates rejection of the null hypothesis at the 5% level.

These results are based on the 5 factor loadings obtained from the MLFA when $n=60$.

Table 4-6
Stationarity Test of Risk-free Rate
from RR Second-step OLS Regression of Arithmetic
Mean Sample Returns on Factor Loadings

Time Period	Stability Test of risk-free rate	Testing RP = .005 per month ^o
----- Panel A: Testing stationarity of risk-free rate when k=1 -----		
III vs. IV (III)	.0001***	.9654
IV vs. III (IV)	.0005***	.0005***
----- Panel B : Testing stationarity of risk-free rate when K=2 -----		
III vs. IV (III)	.0047***	.7371
IV vs. III (IV)	.0042***	.0014***
----- Panel C : Testing stationarity of risk-free rate when k=3 -----		
III vs. IV (III)	.0002***	.7737
IV vs. III (IV)	.0005***	.0002***
----- Panel D : Testing stationarity of risk-free rate when k=4 -----		
III vs. IV (III)	.0006***	.4249
IV vs. III (IV)	.0038***	.0004***
----- Panel E : Testing stationarity of risk-free rate when k=5 -----		
III vs. IV (III)	.0001***	.8923
IV vs. III (IV)	.0004***	.0005***

Note: The numbers represent the probability of F greater than the critical value when Show test is employed.

*** indicates rejection of the null hypothesis at the 1% level.

These results are based on the eigenvectors obtained from the PCA when n=100.

years. However, they did not mention whether the nonstationarity is prevalent for the longer periods.

Summary

The second part of this chapter, following the introduction, briefly reviewed Jobson's multivariate linear regression approach concerned with the test of k-factor APT across assets. The third part of this chapter addressed the issue of the stability of k-factor APT over time. It is shown that, under the null hypothesis of stable k-factor structure over time, realized returns on comparison portfolio within a specific time period can be expressed as a function of its estimated past returns along with the estimated returns on base portfolio for the same time period. The fourth part of this chapter provided some implication of the linear combination of the APT that under certain conditions, any minimum variance portfolio can be expressed as a combination of the arbitrage portfolio obtained from the APT and the minimum variance zero-beta portfolio.

The empirical results with regard to the stability tests of k-factor APT over time reveal that the stability of k-factor structure is violated during the sample periods, indicating that the parsimonious aspects of the APT may not be valid. In addition, related empirical results concerning the stationarity tests of the risk-free rate show that the risk-free rate is stable only for a time period of 6 years, but not for a longer period of 12 years.

Notes

1. If k is large, then it will be difficult to suggest the implications for investment policies. For example, if there are 20 priced factors in the economy, then there will be too much information for the purpose of strategic portfolio planning.
2. The Ross APT relationship holds even under the weaker assumption that the idiosyncratic components of risk are mildly correlated across securities. See Chamberlain (1983a), Chamberlain and Rothschild (1983), Ingersoll (1984), and Grinblatt and Titman (1985) for further discussion of this point.
3. The problem of construction of K mimicking portfolios from a large set of assets has been addressed by Chen (1983). However factor sensitivities from k mimicking portfolios obtained in this manner are not unique, because any orthogonal matrix T such that $TT' = I$ can yield different solution, i.e.,

$$V = B B' + D = (B T)(T' B') + D = B^* B^{*'} + D$$

where V is $(N \times N)$ covariance matrix of $E[(R - E)(R - E)']$,

B is the $(N \times K)$ matrix of factor loadings, and

D is the diagonal matrix of own asset variances.

An alternative way of deriving factor score is presented in Appendix c.

4. The author has benefitted from the discussion with professor Stephen Cosslett for the development of Appendix B.
5. This is simply because the coefficient terms cancel each other. Specifically, since the matrix X and $(I-X)$ looks like as follows,

$$\begin{array}{c} \underline{X} \\ \left[\begin{array}{cccccc} X_{11} & X_{12} & X_{13} & \dots & X_{1n} \\ X_{21} & S_{22} & X_{23} & \dots & X_{2n} \\ \vdots & & & & \vdots \\ X_{n1} & \dots & \dots & \dots & X_{nn} \end{array} \right] \end{array} \quad \begin{array}{c} \underline{I-X} \\ \left[\begin{array}{cccccc} 1-X_{11} & -X_{12} & -X_{13} & \dots & -X_{1n} \\ -X_{21} & 1-X_{22} & -X_{23} & \dots & -X_{2n} \\ \vdots & & & & \vdots \\ -X_{n1} & \dots & \dots & \dots & 1-X_{nn} \end{array} \right] \end{array}$$

Sum of each row vectors in two matrices results in one, which represents the sum of coefficients in each cross-sectional regression.

6. See the GT's definition of "locally efficient" which follows as below.

Definition: Given a subset of S of the set of feasible returns, the return $r \in s$ is said to be locally efficient for s if no other feasible return in the set s with the same r has a lower variance than r

This terminology is used to distinguish these frontiers from the mean-variance efficient frontier of all assets, referred to as the global efficient frontier.

7. The 5-factor structure is selected because the majority of this literature provides evidence that 5 factors are sufficient to explain expected returns.

CHAPTER V
TESTS ON THE ECONOMIC CONTENT OF THE APT

Introduction

Roll et al(1980) have shown that one of the important economic contents of the APT is that no economic/financial variables other than factor loadings play a crucial role in explaining expected security returns. The initial empirical evidence of RR is that the own standard deviation of individual returns does not exert any additional influence once the effect of the factor loadings has been accounted for. Chen (1983) also tested the significance of firm size as well as that of own variance and supported the view of RR by arguing that variables such as own variance and firm size do not contribute additional explanatory power to that of the factor loadings.

On the contrary, however, current empirical evidence on this aspect of the APT has been rather discouraging. Specifically, DFGG (1985a) have obtained the counter-evidence that the unique risk variables such as residual standard deviation along with own standard deviation of stock returns perform as well or better than the factor loadings. Thus, DFGG, by indicating that unique risk is fully as important as common risk, have seriously questioned the empirical methodology of RR and challenged the empirical validity of the APT. Obviously, the empirical evidence with regard to testing the economic content of

the APT is at best mixed. Furthermore, this important issue of testing the economic implication of the APT, nothing other than nondiversifiable risks is priced, has not been fully explored.

In this chapter, further tests are performed to provide a more detailed picture of the relationship between the economic implication of the APT and the stock market anomalies which have not been predictable under the CAPM: the own standard deviation, the impact of higher moment such as skewness, and the firm size effects. In addition, the results concerning the well-known January effect are also discussed.

Methodology

Before discussing in detail the tests of impacts of extraneous economic or financial variables upon the APT, it should be noted that there exist some drawbacks in using maximum likelihood factor analysis (MLFA) used by RR, Chen and DFGG. These problems include nonuniqueness of factor loadings, number of factor or 'priced' factor problem, and indeterminacy of factor scores problem. Therefore, the principal component analysis (PCA) instead of MLFA is utilized because CR (1983) argue that the PCA produces an asymptotically equivalent estimator to the MLFA if there is a strict factor structure, and the PCA yields unique estimates.

To test the economic content of the APT empirically, we can re-specify the APT model with the extraneous variables:

$$E_t \approx RP_o + BRP_k + CX \quad (5-1)$$

where C = a matrix of extraneous variables

X = a vector of the coefficients

In the equation (5-1), if the coefficients vector X is significant, then this can imply that the APT is not consistent in explaining security returns and/or there exist arbitrage opportunities in our economy. Chen (1983) states similarly that

. . . the most important result of the APT is that only those risks that are reflected in the covariance matrix are priced, nothing else. So if we are able to find a variable that is priced even after the factor loadings (FL) are accounted for, the APT would be rejected. (p. 1396)

In this study, matrix C includes the own standard deviation, skewness and firm size proxied by the market value of equities. Specifically, the own standard deviation is selected because it is a diversifiable risk. The residual standard deviation is not included in this study because if adding the own standard deviation along with the residual standard deviation as DFGG (1985) did, then there will be a multicollinearity problem. This is because the residual standard deviation ($D^{1/2}$) is double-counted. The skewness is selected because factor analysis is concerned only with extracting factors from the estimated variance-covariance matrix related to the second moment. However, equation (1-2) alone may not pick up the presence of systematic skewness associated with the third moment of returns (Kraus, 1985).

To state formally, substituting the own standard deviation (SD_i) and/or the skewness term (SK_i) into equation (5-1) results in

$$E_i \approx RP_0 + \sum_j RP_j b_{ij} + RP_{k+1} SD_i \quad (5-2)$$

$$E_i \approx RP_0 + \sum_j RP_j b_{ij} + RP_{k+2} SK_i \quad (5-3)$$

Equations (5-2) and (5-3) are used to examine whether H_0 :

$$RP_{k+p} = 0 \quad (p = 1, 2) \text{ holds.}$$

Recently, the size effect or the complementary small firm effect has attracted wide attention in the academic literature. For example, Banz (1981) and Schwert (1983) reported that small firms systematically experienced average rate of return nearly 20 percent per year greater than those of large firms, even after adjusting for risk. This anomaly, size effect, is observed when portfolios are formed on the basis of firm size. In this regard, a minimum requirement for any alternative model including the APT should be that it eliminates and/or explains the empirical anomalies which arise within the simple CAPM, since unexplained variation may be random and diversifiable. To examine whether the APT eliminates and/or explains the firm size effect, the same logic of equation (5-1) is applied.

$$E_i = RP_0 + \sum_j RP_j b_{ij} + Z_i [\log(MV_i) - \log(MV_m)] \quad (5-4)$$

where MV_i = market value of security i proxied by the product of the price per share and the number of shares outstanding, both measured on the month end.

MV_m = average market value proxy calculated from the entire sample firms.

If this coefficient, Z_i , is statistically insignificant from zero, this would support the notion that only factor risk is priced by the market and the firm size effect is originated from the CAPM misspecification and not from the market inefficiency.

In relation with the size effect, Keim (1983), and Tinic and West (1984) provide the new evidence concerning January effect by employing the two-parameter CAPM. Specifically, Tinic and West document that January is the only month showing a consistently positive, statistically significant relationship between the expected return and the risk premiums. In this respect, it is interesting to investigate whether the factors of the APT are priced only in January. To examine this, RR's cross-sectional OLS procedure is employed to examine the January effect in the APT context.

Empirical Result

The same data of the Chapter IV are utilized for the tests on the economic content of the APT. Thus the monthly returns of the same 60 and 100 randomly selected securities listed on the NYSE from January 1960, to December 1983 period are used.

Before presenting the various results of the tests on the economic content of the APT, it may be helpful to present the RR second-step pricing results. Tables 5-1 through 5-6 report various results obtained from employing RR's second-step regressions. Tables 5-1 and 5-2 present the results of OLS and GLS regressions of arithmetic mean sample returns on the factor loadings for subperiods I, II, III, and IV respectively when the number of assets is 60. From Tables 5-1 and 5-2, it can be seen

Table 5-1
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Factor Loadings obtained from MLFA

Summary Information

Test Period : Jan. 1972 - Dec. 1983
Number of Trading Months : 72 & 144
Number of Securities : 60

K=5	Test Period			
	I(7210-7712)	II(7801-8312)	III(6001-7112)	IV(7201-8312)
RP ₀	0.01266 (2.425)**	0.01358 (2.422)**	.012589 (4.393)**	0.01967 (4.063)**
RP ₁	-0.01103 (-1.316)	0.00452 (0.445)	-.00440 (-.783)	-0.01555 (-1.840)
RP ₂	0.00281 (0.751)	-0.00737 (-2.004)*	-.00257 (-.661)	0.00302 (1.165)
RP ₃	0.00832 (1.830)	0.00647 (1.621)	.00624 (2.292)**	0.00166 (0.518)
RP ₄	-0.00541 (-1.284)	0.00448 (0.921)	.01183 (3.586)**	0.00330 (0.973)
RP ₅	0.00599 (1.024)	0.00433 (0.735)	.00165 (.426)	0.00541 (1.196)
R ²	0.2172	0.1722	.2667	0.1899
$\bar{R}^2$	0.1447	0.0955	.1988	0.1148
F Value	2.996	2.246	3.928	2.531
Prob>F	0.0185**	0.0622*	.0042**	0.0392**
DW	1.763	1.441	2.302	1.593

Note: * significant at a = 10% level

** significant at a = 5% level

The numbers in the parentheses represent the t-value.

Table 5-2
Second-step GLS Regressions of Arithmetic Mean Sample
Returns on the Factor Loadings obtained from MLFA

Summary Information

Test Period : Jan. 1972 - Dec. 1983
Number of Trading Months : 72 & 144
Number of Securities : 60

K=5	Test Period			
	I(7210-7712)	II(7801-8312)	III(6001-7112)	IV(7201-8312)
RP ₀	0.01343 (2.6218)**	0.01396 (2.5179)**	.01252 (4.5213)**	0.02060 (4.3052)**
RP ₁	-0.01228 (-1.4881)	0.00386 (0.3815)	-.00433 (-.7815)	-0.01709 (-2.0532)**
RP ₂	0.00269 (0.7147)	-0.00747 (-2.0588)**	-.00248 (-.6547)	0.00290 (1.1245)
RP ₃	0.00804 (1.7729)	0.00621 (1.5737)	.00627 (2.311)**	0.00174 (0.5421)
RP ₄	-0.00573 (-1.3565)	0.00434 (0.8823)	.01182 (3.5860)**	0.00328 (0.9695)
RP ₅	0.00605 (1.0361)	0.00440 (0.7519)	.00163 (.4185)	0.00515 (0.0453)
R ²	0.2298	0.1740	.2666	0.2030
F Value	3.22	2.27	3.93	2.75
Prob>F	0.0129**	0.0594*	.0042**	0.0275**

Note: ** significant at $\alpha = 5\%$ level

The numbers in the parentheses represent the t-value

Table 5-3
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors obtained from PCA

Summary Information

Test Period : Jan. 1960 - Dec. 1983
Number of Trading Months : 144 & 288
Number of Securities : 100

Coefficients	Test Period		
	III	IV	V
E_0	.00475 (2.534)**	.01236 (6.015)**	.00756 (4.952)**
d_1	.0569 (3.143)**	-.00551 (-.274)	.03463 (2.330)**
d_2	-.00156 (-.208)	.01559 (3.331)**	.01099 (3.826)**
d_3	.00172 (.418)	-.01145 (-2.106)**	-.00060 (-.135)
d_4	-.00512 (-1.161)	-.00339 (-.627)	.00324 (1.119)
d_5	.00839 (1.829)*	-.0082 (-.174)	.00474 (1.606)
R^2	.2209	.1571	.2335
$\bar{R}^2$	.1795	.1123	.1972
Prob>F	.0003 (5.330)	.0061 (3.505)	.0001 (5.727)

Note: The numbers in the parentheses represent the t-value.

* significant at the 10% level

** significant at the 5% level

The numbers in the parentheses in prob>F row represent the F-value.

Table 5-4
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors: Pricing Test for $k=1$ up to $k=5$

Summary Information

Test Period : Time Period III
Number of Trading Months : 144
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(\bar{R}^2)$
Panel A : Based on K=1			
constant	.00495	4.023***	
EV1	.00550	4.447***	.1698(.1613)
Panel B : Based on K=2			
constant	.00553	3.542***	
EV1	.04959	3.252***	
EV2	-.00315	-.606	.1729(.1559)
Panel C : Based on K=3			
constant	.00546	3.455***	
EV1	.05025	3.252***	
EV2	-.00301	-.574	
EV3	.00148	.356	.1740(.1482)
Panel D : Based on K=4			
constant	.00634	3.784***	
EV1	.04188	2.564**	
EV2	-.00483	-.902	
EV3	.00119	-.288	
EV4	-.00660	-1.502	.1932(.1592)
Panel E : Based on K=5			
constant	.00475	2.534**	
EV1	.05694	3.143***	
EV2	-.00156	-.280	
EV3	.00172	.418	
EV4	-.00512	-1.161	
EV5	.00893	1.829*	.2209(.1795)

Note: * significant at 10% level
 ** significant at 5% level
 *** significant at 1% level

Table 5-5
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors: Pricing Test for $k=1$ up to $k=5$

Summary Information

 | Test Period : Time Period IV |
 | Number of Trading Months :144 |
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(\bar{R}^2)$
Panel A : Based on $K=1$			
constant	.01076	6.695***	
EV1	.00969	.603	.0037(-.0065)
Panel B : Based on $K=2$			
constant	.01004	6.539***	
EV1	.01661	1.083	
EV2	.01654	3.534***	.1173(.0991)
Panel C : Based on $K=3$			
constant	.01165	6.821***	
EV1	.00124	.073	
EV2	.01588	3.438***	
EV3	-.01045	-2.025**	.1535(.1271)
Panel D : Based on $K=4$			
constant	.01229	6.135***	
EV1	-.00483	-.246	
EV2	.01562	3.357***	
EV3	-.01135	-2.110**	
EV4	-.00329	-.615	
Panel E : Based on $K=5$			
constant	.01236	6.105***	
EV1	-.00551	-.274	
EV2	.01559	3.331***	
EV3	-.01145	-2.106**	
EV4	-.00339	-.627	
EV5	-.00082	-.174	.1571(.1123)

Note: * significant at 10% level
 ** significant at 5% level
 *** significant at 1% level

Table 5-6
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors: Pricing Test for $k=1$ up to $k=5$

Summary Information

Test Period : Time Period V
Number of Trading Months : 288
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(\bar{R}^2)$
Panel A : Based on K=1			
constant	.00800	7.754***	
EV1	.03045	2.905***	.0815(.0722)
Panel B : Based on K=2			
constant	.00779	8.032***	
EV1	.03250	3.353***	
EV2	.01095	3.802***	.2007(.1842)
Panel C : Based on K=3			
constant	.00842	5.718***	
EV1	.02642	1.840*	
EV2	.01084	3.742***	
EV3	-.00252	-.576	.2034(.1785)
Panel D : Based on K=4			
constant	.00818	5.488***	
EV1	.02874	1.979*	
EV2	.01088	3.758***	
EV3	-.00197	-.449	
EV4	.00305	1.046	.2125(.1793)
Panel E : Based on K=5			
constant	.00756	4.952***	
EV1	.03463	2.330**	
EV2	.01099	3.826***	
EV3	-.00060	-.135	
EV4	.00324	1.119	
EV5	.00474	1.606	.2335(.1927)

Note: * significant at 10% level
 ** significant at 5% level
 *** significant at 1% level

Table 5-7
Summary of Pricing Results

K \ Time Period	III	IV	V
1	1	0	1
2	1	1	2
3	1	2	2
4	1	2	2
5	2	2	2

Note: The numbers represent the number of priced factors.

that zero, one or two factors (for time period III) are priced. However the t-test for the factor loadings obtained from MLFA may not be entirely correct, because the solutions of factor loadings are not unique. Thus, it is not certain that the number of priced factors are changed over time. In this regard, the OLS results of arithmetic mean sample returns on the eigenvectors obtained from PCA when the number of assets is increased up to 100 are reported in Table 5-3. As the eigenvectors obtained from PCA are unique and asymptotically equivalent to the factor loadings, applying t-test to the obtained eigenvectors provides more reliable results than the case of applying t-test to the factor loadings estimated by MLFA.

It is also seen from Table 5-3 that the number of priced factors are, in general, two for subperiods III, IV, and V even though the priced factors are not same each subperiod when the number of factors are fixed at five.

Since the number of factors are fixed at five, we could not be entirely sure whether the correct number of factors are priced and the number of priced factors is constant or variable. Tables 5-4 through 5-6 provide the detailed pricing results when the number of factors is 1 to 5 for subperiods III, IV, and V. Summary results are reported in Table 5-7.

It is illustrated in Tables 5-4 through 5-7 that the number of priced factors is variable when $k=1$ to 4 and constant when $k=5$. Thus, in the case of $k=1$ to 4, this results are consistent with the previous results of cross-sectional MLR-EV tests.

Tables 5-8 through 5-10 report the results when the own standard deviation is added to the equation (5-1). It may be useful to reiterate the equation (5-2).

$$E_i \approx RP_0 + \sum_j RP_j b_{ij} + RP_{k+1} SD_i$$

From the Tables 5-8 through 5-10, it is seen that the coefficient (RP_{k+1}) of own standard deviation is significant for all the cases of $K=1$ to 5 and for subperiods III, IV, and V at 5 percent significance level. This results are inconsistent with the important economic content of the APT that only those risks reflected in the covariance matrix are priced. This finding is not consistent with that of RR (1980) and Chen (1983) that they find no additional explanatory power of own variance to that of the factor loadings. However, this finding is consistent with that of DFGG (1985a, 1985b) in that the own standard deviation seems to be a more important determinant of stock returns than factor-risk premia. The difference of methodology and data is that while RR, Chen, and DFGG employ MLFA and CRSP daily return, this study utilizes PCA and CRSP monthly returns. Of course, although the own standard deviation may not be collinear with the eigenvectors, there is no exact way of judging in which direction the bias will occur when the multicollinearity is intertwined with measurement error in a multiple regression.

Tables 5-11 through 5-13 present the empirical results when the skewness is added to the APT equation. The skewness is important because both MLFA and PCA extract the factors from the estimated variance-covariance matrix. So there is a

Table 5-8
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors and the Own Standard Deviation(OSD)
Summary Information

Test Period:Time Period III(Jan.1960-Dec.1971)
Number of Trading Months :144
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(\bar{R}^2)$
Panel A : Based on K=1			
constant	.00167	1.068	
EV1	.00495	.252	
OSD	.11018	3.174**	.2479(.2324)
Panel B : Based on K=2			
constant	.000525	.243	
EV1	.00660	.333	
EV2	.00423	.771	
OSD	.12263	3.197**	.2525(.2218)
Panel C : Based on K=3			
constant	.000482	.222	
EV1	.00520	.254	
EV2	.00426	.773	
EV3	-.00122	-.299	
OSD	.12510	3.174**	.2532(.2218)
Panel D : Based on K=4			
constant	.00065	.246	
EV1	.00535	.259	
EV2	.00398	.654	
EV3	-.00119	-.291	
EV4**	-.00054	-.113	
OSD	.12277	2.752**	.2533(.2167)
Panel E : Based on K=5			
constant	.00029	.110	
EV1	.01947	.816	
EV2	.00505	.822	
EV3	-.00057	-.138	
EV4	-.00031	-.064	
EV5	.00544	1.168	
OSD	.10813	2.338**	.2641(.2167)

Table 5-9
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors and the Own Standard Deviation(OSD)
Summary Information

Test Period:Time Period IV (Jan.1972-Dec.1983)
Number of Trading Months :144
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(\bar{R}^2)$
Panel A : Based on K=1			
constant	.00623	3.214**	
EV1	-.06479	-2.662**	
OSD	.13594	3.889**	.1381(.1203)
Panel B : Based on K=2			
constant	.00692	3.653**	
EV1	-.04105	-1.559	
EV2	.01081	2.150**	
OSD	.10087	2.655**	.1777(.1520)
Panel C : Based on K=3			
constant	.00854	3.984**	
EV1	-.04612	-1.752*	
EV2	.01098	2.200**	
EV3	-.00810	-1.573	
OSD	2.311	2.311**	.1986(.1648)
Panel D : Based on K=4			
constant	.00921	3.889**	
EV1	-.05285	-1.871	
EV2	.01068	2.124**	
EV3	-.00905	-1.689*	
EV4**	-.00351	-.671	
OSD	.0894	2.316**	.2024(.1600)
Panel E : Based on K=5			
constant	.00931	3.907**	
EV1	-.05722	-1.952*	
EV2	.01036	2.043**	
EV3	-.00928	-1.722*	
EV4	-.00385	-.728	
EV5	-.00271	-.583	
OSD	.09330	2.374**	.2053(.1540)

Table 5-10
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors and the Own Standard Deviation(OSD)
Summary Information

Test Period: Time Period V (Jan.1960-Dec.1983)
Number of Trading Months :288
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(R^2)$
Panel A : Based on K=1			
constant	.00363	3.315**	
EV1	-.04518	-3.114**	
OSD	.14582	6.497**	.3601(.3469)
Panel B : Based on K=2			
constant	.00406	3.639**	
EV1	-.03530	-2.271**	
EV2	.00471	1.672*	
OSD	.12847	5.235**	.3782(.3587)
Panel C : Based on K=3			
constant	.00338	2.079**	
EV1	-.03152	-1.864*	
EV2	.00465	1.642	
EV3	.00230	.578	
OSD	.13186	5.208**	.3803(.3542)
Panel D : Based on K=4			
constant	.00245	1.496	
EV1	-.03214	-1.944*	
EV2	.00418	1.506	
EV3	.00379	.962	
EV4**	.00600	2.323**	
OSD	.14364	5.685**	.4140(.3828)
Panel E : Based on K=5			
constant	.00243	1.476	
EV1	-.03501	-1.902*	
EV2	.00398	1.399	
EV3	.00365	.917	
EV4	.00604	2.325**	
EV5	-.0010	-.362	
OSD	.1474	5.368**	.4148(.3770)

Table 5-11
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors and the Skewness (SK)
Summary Information

Test Period: Time Period III (Jan. 1960-Dec. 1971)
Number of Trading Months : 144
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(R^2)$
Panel A : Based on K=1			
constant	.00591	4.652***	
EV1	.05542	4.610***	
SK	-.00333	-2.350***	.2145(.1983)
Panel B : Based on K=2			
constant	.00657	4.140***	
EV1	.04926	3.301***	
EV2	-.00357	-.701	
SK	-.00336	-2.366**	.2185(.1941)
Panel C : Based on K=3			
constant	.00650	4.049***	
EV1	.04987	3.303***	
EV2	-.00344	-.670	
EV3	.00137	.337	
SK	-.00336	-2.351**	.2194(.1866)
Panel D : Based on K=4			
constant	.00799	4.644***	
EV1	.03788	2.394**	
EV2	-.00610	-1.178	
EV3	.00094	.234	
EV4**	-.00938	-2.156**	
SK	-.00406	-2.822***	.2562(.2167)
Panel E : Based on K=5			
constant	.00647	3.375***	
EV1	.05176	2.938***	
EV2	-.00308	-.568	
EV3	.00142	.358	
EV4	-.00793	-1.808*	
EV5	.00765	1.720*	
SK	-.00391	-2.741***	.2791(.2326)

Table 5-12
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors and the Skewness (SK)
Summary Information

Test Period: Time Period IV (Jan.1972-Dec.1983)
Number of Trading Months : 144
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(R^2)$
Panel A : Based on K=1			
constant	.01046	6.113***	
EV1	.00888	.548	
SK	.00077	.533	.0066(-.0139)
Panel B : Based on K=2			
constant	.00968	5.932***	
EV1	.01571	1.017	
EV2	.01661	3.538***	
SK	.00088	.649	.1212(.0937)
Panel C : Based on K=3			
constant	.01144	6.184***	
EV1	.00123	.073	
EV2	.01593	3.431***	
EV3	-.01017	-1.930*	
SK	.00042	.305	.1543(.1187)
Panel D : Based on K=4			
constant	.01207	5.778***	
EV1	-.00534	-.270	
EV2	.01566	3.351***	
EV3	-.01106	-2.027**	
EV4	-.0035	-.657	
SK	.00053	.386	.1582(.1134)
Panel E : Based on K=5			
constant	.01215	5.693***	
EV1	-.00613	-.303	
EV2	.01563	3.325***	
EV3	-.01117	-2.026	
EV4	-.00368	-.671	
EV5	-.00093	-.197	
SK	.00055	.396	.1586(.1043)

Table 5-13
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors and the Skewness (SK)
Summary Information

Test Period: Time Period V (Jan.1960-Dec.1983)
Number of Trading Months : 288
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(\bar{R}^2)$
Panel A : Based on K=1			
constant	.00836	7.458***	
EV1	.03095	2.988***	
SK	-.00092	-.824	.0879(.0691)
Panel B : Based on K=2			
constant	.00819	7.796***	
EV1	.03309	3.407***	
EV2	.01103	3.829***	
SK	-.00104	-.992	.2088(.1840)
Panel C : Based on K=3			
constant	.00888	5.766***	
EV1	.02659	1.852*	
EV2	.01092	3.768***	
EV3	-.00269	-.616	
SK	-.00107	-1.013	.2119(.1787)
Panel D : Based on K=4			
constant	.00857	5.331***	
EV1	.0282	1.936*	
EV2	.01092	3.761***	
EV3	-.00224	-1.507	
EV4**	.00224	.704	
SK	-.00075	-.654	.2160(.1743)
Panel E : Based on K=5			
constant	.00797	4.873***	
EV1	.03417	2.290**	
EV2	.01103	3.831***	
EV3	-.00087	-.195	
EV4	.00237	.752	
EV5	.00480	1.621	
SK	-.00080	-.706	.2376(.1884)

possibility that the APT equation (1-2) alone may not pick up the presence of systematic skewness.

From the table 4-17, the skewness, estimated by

$$\frac{[E(R_i - ER_i)]^3}{\sigma^3}$$

turns out to be important in subperiod III because the coefficients of skewness are all significant for $K=1$ to 5 and the signs of coefficients are negative. This results are consistent with the prediction of researchers who believe that the investors prefer positive skewness.

However these phenomena are no longer persistent in subperiod IV and the entire sample period V. From the Table 5-13, it can be seen that the coefficients of skewness are not significant for any number of k , and the signs of each coefficients are positive for subperiod IV, but negative for entire sample period I. Hence, as in the case of the number of priced factors, the explanatory power of skewness may not be stable over time. One possible conjecture is that since subperiod IV includes the event of oil embargo, and con.ed. announcement, this period may contain the certain period of instability (Pettway, 1978).

Tables 5-14 through 5-16 report the empirical results when the firm size, proxied by the market value of equity, is added to the APT equation to examine whether the size effect prevalent in the CAPM is also persistent in the multi-factor APT model.

Table 5-14
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors and the Firm Size (FS)
Summary Information

Test Period: Time Period III (Jan. 1960-Dec. 1971)
Number of Trading Months : 144
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	R^2 ($\bar{R}^2$)
Panel A : Based on K=1			
constant	.00464	3.200***	
EV1	.06229	4.217***	
FS	.00030	1.074	.2380 (.2138)
Panel B : Based on K=2			
constant	.00660	3.533***	
EV1	.04290	2.299**	
EV2	-.00930	-1.633	
FS	.00027	.994	.2695 (.2841)
Panel C : Based on K=3			
constant	.00679	3.564***	
EV1	.04091	2.146**	
EV2	-.01013	-1.718*	
EV3	-.00293	-.590	
FS	.00028	1.008	.2736 (.2260)
Panel D : Based on K=4			
constant	.00782	4.097***	
EV1	.03186	1.680*	
EV2	-.01334	-2.258**	
EV3	-.00495	-1.010	
EV4**	-.01012	-2.180**	
FS	.00021	.789	.3269 (.2708)
Panel E : Based on K=5			
constant	.00616	3.162***	
EV1	.04572	2.408**	
EV2	-.00996	-1.712*	
EV3	-.00435	-.925	
EV4	-.00684	-1.476	
EV5	.01145	2.516**	
FS	.00021	.800	.3922 (.3304)

Table 5-15
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors and the Firm Size (FS)
Summary Information

Test Period: Time Period IV (Jan.1972-Dec.1983)
Number of Trading Months :144
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(\bar{R}^2)$
Panel A : Based on K=1			
constant	.01050	5.720***	
EV1	.01525	.827	
FS	.00018	.698	.0185(-.0099)
Panel B : Based on K=2			
constant	.01023	5.860***	
EV1	.01538	.878	
EV2	.01409	2.914***	
FS	.00009	.338	.1275(.0890)
Panel C : Based on K=3			
constant	.01099	5.303***	
EV1	.00841	.415	
EV2	.01381	2.836***	
EV3	-.00417	-.688	
FS	.00011	.445	.1336(.0819)
Panel D : Based on K=4			
constant	.01165	5.131***	
EV1	.00217	.098	
EV2	.01342	2.729***	
EV3	-.00501	-.809	
EV4**	-.00438	-.724	
FS	.00009	.356	.1404(.0753)
Panel E : Based on K=5			
constant	.01305	5.673***	
EV1	-.01144	-.500	
EV2	.01197	2.479**	
EV3	-.0062	-1.027	
EV4	-.00749	-1.237	
EV5	-.01313	-2.179**	
FS	.00015	.600	.1989(.1250)

Table 5-16
Second-step OLS Regressions of Arithmetic Mean Sample
Returns on the Eigenvectors and the Firm Size (FS)
Summary Information

Test Period: Time Period IV (Jan.1960-Dec.1983)
Number of Trading Months : 288
Number of Securities : 100

Variable	Estimated Coefficient	t ratio	$R^2(\bar{R}^2)$
Panel A : Based on K=1			
constant	.00827	6.954***	
EV1	.03178	2.708***	
FS	.00025	1.507	.1022(.1926)
Panel B : Based on K=2			
constant	.00806	7.215***	
EV1	.03255	2.958***	
EV2	.01091	3.585***	
FS	.00018	1.130	.2201(.1895)
Panel C : Based on K=3			
constant	.00903	5.555***	
EV1	.02339	1.493	
EV2	.01067	3.483***	
EV3	-.00390	-.822	
FS	.00018	1.162	.2264(.1895)
Panel D : Based on K=4			
constant	.00878	5.301***	
EV1	.02548	1.601	
EV2	.01070	3.487***	
EV3	-.00344	-.717	
EV4**	.00250	.804	
FS	.00017	1.029	.2324(.1861)
Panel E : Based on K=5			
constant	.00810	4.758***	
EV1	.03159	1.941*	
EV2	.01095	3.591***	
EV3	-.00189	-.388	
EV4	.00282	.913	
EV5	.00481	1.543	
FS	.00015	.914	.2540(.1994)

While the firm size effect was initially documented by Banz (1981) and Reinganum (1981), voluminous studies provide evidence that small firms had higher average returns than large firms even after adjusting for risk via the CAPM. Accordingly, this evidence is considered as an important anomaly and a rejection of the joint hypothesis that the CAPM is valid and the capital market is efficient.

Applying the APT model, this section finds an interesting results that no single coefficient of the proxy of firm size variable is significant for all K from 1 to 5 and over all sample periods. This finding is essentially an equivalent evidence of Chen (1983) who found that the firm size effect is captured by the factor loadings of the APT. Thus, the results of this study are consistent with the hypothesis that risk is the explanation for the firm size effect and that the capital market is efficient. However, since the samples used in this study are obtained only from the monthly CRSP tape, it is not entirely certain that the returns of small firms whose stocks are traded in the over-the-counter market are also subject to the same finding of this study.

For the generalization of the results of this study to the small firms whose stocks are traded in OTC market, larger scale tests should be followed. However, it is beyond the scope of this study.

In conjunction with the firm size effect, Keim (1983) points out that more than 50 percent of the magnitude of the size effect is due to anomalous January returns. Tinic and West (1984) also found evidence of the seasonality that the

relationship between returns and systematic risk is consistently positive only in January. For the remaining eleven months of the year, they further showed, the risk premia estimated by two-parameter CAPM, on average, are not significantly different from zero.

An interesting issue associated with this study is whether the factors of the APT are also priced only in January. If the seasonalities in the estimated risk premium are not significant in January, then the January effect found by using two-parameter CAPM could be simply due to the CAPM (empirically market model) misspecification.

Tables 5-17 through 5-19 present the empirical results of cross-sectional regression of observed returns on the largest five eigenvectors for January. It is seen that two to four factors are frequently priced and one and five factor are occasionally priced in January.

However, these results are also true for the remaining eleven months (the results of remaining eleven months are not reported for the purpose of conserving space). Using 5-eigenvector model, it is easy to see that there is nothing unique in January. The only notable exception is that January of 1967 in subperiod III, January of 1975, 1976, and 1977 in subperiod IV, and January of 1966, 1971, 1972, 1975, and 1977 have the highest R^2 among 12 months.

This finding could be related to the oil embargo, however, that should wait for the detailed test. Based upon the finding of this chapter, it is quite conceivable that the firm size

Table 5-17
Cross-sectional Regression of Observed Returns
on the Largest Five Eigenvectors for the January Effect

Summary Information

Test Period : Time Period III
Number of Trading Months : 144
Number of Securities : 100

Time	E_o	d_1	d_2	d_3
6001	.01164 (.46)	-.6352 (-2.57)**	-.1131 (-1.48)	-.2077 (-3.70)***
6101	-.0530 (-2.10)**	1.1896 (4.88)***	.2610 (3.48)***	.2460 (4.45)***
6201	-.0431 (-1.58)	.2269 (.86)	.0477 (-.59)	.2746 (4.59)***
6301	.0280 (1.18)	.3781 (1.64)	.0654 (.92)	.1457 (2.79)***
6401	-.0281 (-1.28)	.4370 (2.05)**	.1126 (1.71)*	.1275 (2.64)***
6501	.0108 (.46)	.4198 (1.84)*	.0252 (.36)	-.1873 (-3.62)***
6601	.0009 (.03)	.4084 (1.52)	-.2804 (-3.40)**	.1101 (1.81)*
6701	.06210 (2.25)**	-.5578 (2.09)**	-.1481 (-1.80)*	-.1769 (-2.92)***
6801	.0769 (2.73)***	-.9979 (-3.66)**	.0691 (.82)	.1035 (1.68)*
6901	-.0028 (-.13)	-.1150 (-.55)	-.0365 (-.57)	-.0057 (-.12)
7001	-.0015 (-.05)	-.7484 (-2.61)**	.0485 (.55)	-.0408 (-.63)
7101	-.0032 (-.11)	.8214 (2.79)***	-.2618 (2.89)***	-.0247 (-.37)

Note: * significant at 10% level
** significant at 5% level
*** significant at 1% level

Table 5-17 (continued)

Time	d_4	d_5	R^2
6001	-.708 (-2.83)***	-.0651 (-1.04)	.1901 (.1470)
6101	.0544 (.92)	.1353 (2.19)**	.2947 (.2574)
6201	.1250 (1.94)*	-.0040 (-.06)	.2232 (.1819)
6301	-.1495 (-2.67)***	.0242 (.42)	.1916 (.1486)
6401	-.0243 (.47)	.0686 (1.27)	.0982 (.0502)
6501	-.0264 (-.47)	-.1123 (-1.94)*	.2479 (.2079)
6601	-.1798 (-1.22)	.0788 (1.16)	.3196 (.2834)
6701	-.4131 (-6.35)***	-.2735 (-4.04)***	.5626 (.5393)
6801	-.3755 (-5.66)***	.0687 (1.00)	.4207 (.3899)
6901	.0857 (1.68)*	.0657 (1.24)	.0723 (.0230)
7001	-.1763 (-2.52)**	.0550 (.76)	.2034 (.1600)
7101	.1720 (2.40)**	.0644 (.86)	.3547 (.3204)

Note: The numbers in parentheses represent t-value
 Only January results are presented in the interest of
 conserving space

Table 5-18
Cross-sectional Regression of Observed Returns
on the Largest Five Eigenvectors for 'January' Effect

Summary Information

Test Period : Time Period IV
Number of Trading Months : 144
Number of Securities : 100

Time	E_o	d_1	d_2	d_3
7201	.0532 (1.60)	-.1888 (-.58)	.1400 (1.84)*	-.2757 (-3.12)***
7301	.02244 (.84)	-.7182 (-2.74)***	.3818 (6.26)***	-.0816 (-1.15)
7401	.04698 (1.12)	.03865 (.10)	-.4648 (-5.25)***	-.3338 (-3.24)***
7501	-.0343 (-1.03)	2.132 (6.57)***	-.888 (-11.76)***	-.828 (-.94)
7601	.0063 (.19)	1.490 (4.49)***	-.0226 (-.28)	-.2543 (-2.83)***
7701	.0249 (1.24)	-.5419 (-2.75)***	-.0528 (-1.15)	-.4140 (-7.76)***
7801	-.0262 (-1.57)	-.2882 (-1.77)*	-.0468 (-1.23)	-.1589 (-3.60)***
7901	.0922 (3.23)***	-.1113 (-.398)	.0841 (1.29)	-.5371 (-7.10)***
8001	-.0058 (-.17)	.7644 (2.33)**	.3674 (4.81)***	-.5863 (-6.61)
8101	-.0066 (-.253)	-.0858 (-.339)	-.4927 (-8.36)***	.0388 (.567)
8201	.0414 (1.12)	-.5133 (-1.42)	-.3545 (-4.22)***	-.0799 (-.82)
8301	.0445 (1.10)	.1051 (.27)	.2614 (2.83)***	-.2262 (-2.11)**

Table 5-18 (continued)

Time	d_4	d_5	R^2
7201	-.3749 (-4.27)***	.0808 (1.06)	.2859 (.2479)
7301	.1488 (2.11)**	.1501 (2.45)**	.4588 (.4300)
7401	-.0197 (-.19)	-.0200 (-.23)	.3151 (.2787)
7501	.573 (6.56)***	.1364 (1.80)*	.7312 (.7169)
7601	-.0449 (-.50)	-.0233 (-.30)	.4220 (.3913)
7701	-.2201 (4.15)***	-.0811 (-1.76)*	.5347 (.5100)
7801	-.0665 (-1.51)	.1539 (4.03)***	.2677 (.2287)
7901	-.1146 (-1.52)	.1124 (1.72)*	.4363 (.4063)
8001	.0489 (.55)	.1570 (2.05)**	.5449 (.5207)
8101	.0273 (.400)	.1692 (2.86)***	.4643 (.4358)
8201	-.1513 (-1.56)	.0821 (.97)	.1814 (.1378)
8301	-.1186 (-1.11)	-.1375 (-1.48)	.1711 (.1270)

Table 5-19
Cross-sectional Regression of Observed Returns
on the Largest Five Eigenvectors for 'January' Effect

Summary Information

Test Period : Time Period V
Number of Trading Months : 288
Number of Securities : 100

Time	E_o	d_1	d_2	d_3
6001	.0202 (.65)	-.7310 (-2.40)**	-.0468 (-.79)	-.1356 (-1.49)
6101	.0146 (.47)	.5348 (1.76)*	.1423 (2.42)**	.1098 (1.21)
6201	.0104 (.31)	-.2849 (-.87)	.1209 (1.90)*	-.1583 (-1.61)
6301	.0003 (.011)	.6516 (2.36)*	.0309 (.58)	.0888 (1.08)
6401	.0330 (1.27)	-.1575 (-.62)	.1063 (2.17)**	-.0494 (-.65)
6501	.0333 (1.21)	.1800 (.67)	.0357 (.69)	-.0534 (-.67)
6601	-.0254 (-.85)	.6662 (2.30)**	.2464 (4.39)***	-.2195 (-2.53)**
6701	-.0038 (-.10)	1.1847 (3.27)***	-.0579 (-.83)	-.1822 (-1.68)*
6801	.0658 (1.84)*	-.8892 (-2.56)**	.0392 (.58)	.0818 (.79)
6901	-.0072 (-.29)	-.0695 (-.29)	-.0368 (-.80)	.0141 (.20)
7001	.0239 (.70)	-1.000 (-3.02)***	.1462 (2.29)**	-.0206 (-.21)
7101	.0428 (1.21)	.3706 (1.08)	-.0346 (-.52)	-.3333 (-3.24)***

Table 5-19 (continued)

Time	d_4	d_5	R^2
6001	-.0595 (-1.00)	-.1401 (-2.31)**	.0952 (.0471)
6101	-.1028 (-1.74)*	-.1595 (-2.64)***	.1932 (.1503)
6201	-.1471 (-2.29)**	-.0214 (-.33)	.1096 (.0622)
6301	-.1384 (-2.57)**	.0375 (.68)	.1430 (.0974)
6401	.0043 (.09)	-.0607 (-1.21)	.0666 (.0169)
6501	.1262 (2.41)**	-.2187 (-4.10)***	.2338 (.1931)
6601	.0073 (.13)	.0897 (1.56)	.4120 (.3807)
6701	-.1510 (-2.14)**	-.1213 (-1.68)*	.4064 (.3748)
6801	-.3012 (-4.45)***	.0063 (.09)	.3068 (.2699)
6901	.0968 (2.08)**	.1065 (2.24)**	.1077 (.0603)
7001	.0045 (.07)	-.0745 (-1.13)	.2158 (.1741)
7101	.2233 (3.32)***	-.0489 (-.71)	.3490 (.3143)

Table 5-19 (continued)

Time	E _o	d ₁	d ₂	d ₃
7201	.1047 (3.02)***	-.6873 (-2.04)**	.1753 (2.69)***	-.5988 (-5.93)***
7301	.0429 (1.33)	-.9160 (-2.92)***	.3942 (6.50)***	.0417 (.45)
7401	.0699 (1.49)	-.2170 (-.48)	-.4295 (-4.86)***	-.2108 (-1.54)
7501	-.0237 (-.53)	1.9936 (4.53)***	-.9102 (-10.71)***	.3591 (2.73)***
7601	.0431 (1.02)	1.1159 (2.72)***	-.0528 (-.67)	-.2395 (-1.95)*
7701	.0149 (.58)	-.4523 (-1.79)*	-.0247 (-.51)	.0089 (.12)
7801	-.0259 (-1.26)	-.3004 (-1.50)	-.0241 (-.62)	-.0886 (-1.48)
7901	.0636 (1.79)*	.1556 (.451)	.1337 (2.01)**	-.1849 (-1.79)*
8001	-.0255 (-.59)	.9462 (2.25)**	.3752 (4.16)***	-.1429 (-1.13)
8101	-.0476 (-1.49)	.2952 (.96)	-.4816 (-8.11)***	.1943 (2.12)**
8201	.0431 (.98)	-.5257 (-1.23)	-.3656 (-4.44)***	-.1510 (-1.18)
8301	.0508 (1.08)	.0644 (.14)	.2932 (3.31)***	-.2792 (-2.04)**

Table 5-19 (continued)

Time	d_4	d_5	R^2
7201	-.0716 (-1.09)	.2717 (4.03)***	.4606 (.4319)
7301	-.1090 (-1.78)*	.0979 (1.57)	.4523 (.4231)
7401	-.2800 (-3.14)***	.1729 (1.90)*	.3028 (.2657)
7501	-.3969 (-4.63)***	.1308 (1.50)	.6513 (.6328)
7601	-.2447 (-3.06)***	.0694 (.85)	.3779 (.3448)
7701	-.4045 (-8.21)***	-.1545 (-3.08)	.4609 (.4322)
7801	-.1084 (-2.77)***	.145 (3.64)***	.2204 (.1789)
7901	-.4496 (-6.69)***	.0261 (.38)	.3947 (.3625)
8001	-.4742 (-5.77)***	.1651 (1.97)*	.4705 (.4423)
8101	-.0228 (-.38)	.0797 (1.31)	.4423 (.4136)
8201	.0096 (.12)	-.1405 (-1.66)	.1934 (.1505)
8301	-.1148 (-1.28)	-.1546 (-1.70)*	.2188 (.1773)

effect or its cousin, January effect could be simply due to the misspecification of market model. Or the magnitude of the firm size and January effect is overestimated.

Recently, the first point, the misspecification of market model is reported by Huang and Jo (HJ) (1986). A few of previous studies are concerned with the misspecification of the CAPM, however, no study performs a formal test of the misspecification of the market model. Huang et al, by utilizing tests proposed by White (1980) and Hausman(1978), conclude that misspecification as well as heteroscedasticity associated with market model (not with the CAPM) is, in general, a serious problem. Another possibility, the overestimation of the magnitude of the firm size effect and/or January effect, is studied by Chan, Chen, and Hsieh (CCH) (1985). Chan et al, by employing a multi-factor pricing model, reported that the risk-adjusted difference in returns between the top five percent and the bottom five percent of the NYSE firms is about one to two percent a year, which is a tremendous drop from about 20 percent difference per year between small firms and large firms reported by previous studies using the CAPM. They also found that January does not reveal any particular pattern.

All in all, we could not reject the joint hypothesis that the capital markets are efficient and that five-eigenvector APT model is an equilibrium pricing model, since neither firm size variable nor January pricing results provides a particular systematic pattern.

Summary

This chapter presented a variety of evidence on the tests of economic content of the APT that no economic/financial variables other than factor loadings play a crucial role in explaining expected security returns. By utilizing the principal component analysis (PCA) suggested by CR (1983), this chapter investigated whether the other financial variables including the own standard deviation, the skewness, and the firm size had additional explanatory power after the adjustment of eigenvectors. Also, the related January effect is examined whether it is persistent in the APT model as in the CAPM. It turns out that all financial variables except the own standard deviation do not have additional explanatory power after adjusting risk by k-eigenvectors, which is consistent with the prediction of the APT. However, the strange phenomena happen when the own standard deviation (OSD) is added to the APT equation such that the coefficient of OSD is persistently significant for all the cases of $k=1$ up to 5 and for entire sample period at 5 percent significance level, which is inconsistent with the prediction of the APT. However, this finding is consistent with that of DFGG (1985a, 1985b) in the sense that own standard deviation seems to be a more important determinant of stock returns than factor-risk premia.

CHAPTER VI CONCLUSIONS

The purpose of this study was to enhance the current understanding of the Arbitrage Pricing Theory (APT) and perform various tests to examine whether the APT is a parsimonious theory and an empirically tractable model.

In this dissertation, a number of new tests on the empirical relevance of the APT are derived and performed to address more completely the basic and fundamental questions of convergence, stationarity, and economic content of the APT. Specifically, a new approach is taken to examine whether the variance of idiosyncratic risk becomes bounded. Also, the question of stationarity is addressed to investigate whether the risk measures from one period can explain returns in another.

In addition, the economic content of the APT is scrutinized to determine if only those risks reflected in the covariance matrix are priced so that no economic/financial variables other than eigenvectors play a crucial role in explaining expected security returns.

In particular, to test the boundary condition of diversifiable risks, this paper derived necessary and sufficient conditions of minimizing the differences of factor pattern matrices, between the two factor structures, the strict factor

structure and the approximate factor structure and examined the convergence between two factor structures.

Employing the orthogonal procrustes rotation methodology (OPRM), this study found evidence that the average squared differences of two factor structures of two subperiods and entire sample periods are getting smaller when the number of assets grows. These findings are consistent with the premise that the two factor structures should converge as the number of assets becomes large, implying the boundedness of diversifiable risks. The results obtained from utilizing the well-known varimax rotation methodology (VRM) generally are supportive of the convergence. These findings are particularly interesting because they conform to the economic intuition that the variance of idiosyncratic risks is bounded and so should not affect the pricing.

Moreover, by examining the convergence of two factor structure, this study refutes the argument of Shanken (1982). In particular, Shanken challenges the view of Ross by arguing that the boundary condition of residual variance condition of the APT is untestable, since the variance obtained from finite data should be finite. However, this paper shows that even by using a finite sample, it is possible to examine the boundedness of the variance of diversifiable risks.

In the later part of Chapter III, some evidence on the number of factors and eigenvalues is presented. This can be briefly summarized. First, although the first eigenvalue dominates others for all sample periods, two to four factors

extracted from the return generating process are significant in explaining security returns. Second and more importantly, the evidence implies that both the number of factors and priced factors can be variable instead of constant. If the number of priced factors are changing over time, the parsimonious aspects of the APT are weakened and practical applicability of the APT is limited. This important issue of the stability of k-factor APT is more thoroughly investigated in Chapter IV. It is shown that, under the null hypothesis of stable k-factor structure over time, realized returns on a comparison portfolio within a specific time period can be expressed as a function of its historical expected returns and the expected returns of a reference portfolio at the same point in time. The empirical results reveal that the stability of k-factor structure is violated during the sample periods, indicating that the parsimonious aspects of the APT are weakened. Also, related empirical results concerning the stationarity tests of the risk-free rate as well as tests for the equality of intercepts to the risk-free rate show that the risk-free rate is stable for a time period of 6 years, but not for a longer period of 12 years.

Chapter V presented a variety of evidence on the economic content of the APT. Specifically, the crucial economic content of the APT is that once the effects of the eigenvectors (or factor loadings) are accounted for, there are no other variables which exert any additional influence on the expected returns. The presence of any such influence would mean a rejection of the APT. It turns out that the extraneous variables including firm size, and skewness do not exert any additional explanatory power

after the adjustment of the systematic risk. Also, the related January effect does not exhibit any particular systematic pattern which can distinguish it from the remaining eleven months. However, when its own standard deviation (OSD) is added to the APT equation, the strange phenomenon occurs that the coefficient of the OSD is persistently significant for all cases of $k=1$ up to 5 for all sample periods. This finding is consistent with that of DFGG (1985a, 1985b) in the sense that the OSD seems to be a more important determinant of stock returns than factor-risk premia.

The results suggest that in the majority of the cases, the extraneous variables excluding the OSD do not have additional influence after the expected returns have been adjusted for systematic risks. However, the empirical results of cross-sectional regression with the OSD cast some doubt as to the validity of the assertion that only those risks reflected in the covariance matrix are priced.

Overall, this study evaluates the APT model as a theoretically sound one, but practically limited. Hence, the message of this paper to the parties of interest including the general investors in the private sector as well as policymakers in the public sector is that they should be very cautious when applying the APT to financial management decisions.

APPENDIX A
A LITERATURE SURVEY OF K-FACTOR APT

Author (publication)	Statistical or theoretical methodology	the number of factors concluded or used
Gehr (JMWF 78)	Factor Analysis by a varimax pre-rotation a promax final rotation	apparently 2, or at most 3 Rule: Eigenvalue>1
Roll & Ross (JF 12/80)	Maximum Likelihood Factor Analysis (MLFA)	at least 3 and probably 4 'priced' factors
Pari (UPD*, Virginia Polytech, 80)	common factor analysis and principal factor method	4 common factors (4 homogeneous industry clusters)
Reinganum (JF 9/81)	Ranking securities on the basis of market value of the common stocks and get mean excess return of the 10 market value (MV) port- folio	3~5 factors following by RR (found evidence inconsistent with APT)
Chen (UPD, UCLA 81)	MLFA	5 were selected based upon RR and Reinganum
Fogler, Kose, and Tipton (JF 5/81)	principal component analysis $r_{it} = B_{i0} + B_{is} S_t + B_{ig} g_t + B_{ic} C_t + e_{it}$	3 factors returns for a stock .U.S. Government Bond Index (g_t) .corporate bond Index (C_t)(hypothesized)

Author (publication)	Statistical or theoretical methodology	the number of factors concluded or used
Oldfield & Rogalski (JF 5/81)	Maximum Likelihood procedure .Get the minimum variance zero beta portfolio	5 factors used to minimize the residual variation
Fogler (JPM Summer 82)	Principal Component Analysis	3 indexes overall market, interest sensitive, & energy sensitive
Berges (UPD, Purdue 82)	Simulation F.A.	at least 3 factors 5 factors used
Kryzanowski and To (JFQA 83)	Rao's (or Canonical) factor analysis and alpha factor analysis Scree and discontinuity test	5 factors hypothesized # of factors = $k = f(n)$ (+) $k = f(T)$ (-)
Elton, Gruber, & Rentzler (JF 83)	Theoretical $\bar{R}_i = \bar{R}_Z + B_{1i}(R'_m - \bar{R}_Z)$ $+ B_{2i}(\Phi - \bar{\Phi}) + e_i$	(1) Both theory and empirical evidence support that the <u>in-</u> <u>flation factor</u> has an independent effect on the returns on differe assets beyond the mark impact (2) Another implica- tion is that the usual two fund separation theorem is replaced by a three fund separation theorem

Author (publication)	Statistical or theoretical methodology	the number of factors concluded or used
Brown & Weinstein (JF 6/83)	The bilinear paradigm (theory) MLFA(empirical)	as few as 3 economy- wide factors, and no more than 5 if the APM is correct
Gibbons (Working Paper, Stanford 7/83)	.Likelihood Ratio Test .Box (1949) Transformation	.6 to 9 factors .included bonds along with stocks
Chamberlain & Rothschild (Econometrica 9/83)	Hilbert space setting (theory)	approximate 1-factor structure (theoretical result)
Chen, Roll, and Ross (WP** 12/83)	Constructed the hypothetical economic factors	industrial production, risk premium, excess returns on long bonds and measures of unan- ticipated inflation
Chen (JF 12/83)	MLFA	5 (based Upon RR)
Cho, Elton & Gruber (JFQA 3/84)	simulation, MLFA	2 factors for most of the time, occasionally 3 factors are needed
Jordan (UPD, UF, 84)	.MLFA Applied to Event Study .zero beta portfolio	.6 factors # of factors = $f(\text{size},$ criterion)
Dhrymes, Friend, & Gultekin (JF 6/84)	.Econometrically showed the identification problem .MLFA	$k = f(n)$ (+)

Author (publication)	Statistical or theoretical methodology	the number of factors concluded or used
Roll & Ross (FAJ 6/84)	Descriptive Essay	4 factors inflation, industrial production, risk pre- mium & the slope of the term structure of interest rates (based on CRR WP)
Pari & Chen (JFR Summer 84)	Rao's canonical factor analysis	3 factors market index, price volatility of energy, & measure of interest- rate risk
Dhrymes (JPM Summer 84)	.MLFA	the smaller the sample the smaller the number of factors
Bower, Bower & Logue (JF 9/84)	.FA & Regression .Theil measure(U^2) of the sum of squared difference .applied to utility stock	.Used 4 factors .APT is superior to CAPM through time and across assets
Jo (WP,UF 11/84)	Common FA MLFA Eigenvalue test Scree test	at least 2 factors to 5 or 6 factors .# of factors = f(sample, criterion)
Cho (JF 12/84)	Inter-battery FA	.5 or 6 inter-group common factors (IGF) .IGF#f(size of groups)
Gultekin & Rogalski (JF 3/85)	.zero beta portfolio .Applied to U.S. Treasury Security Market .Compare the empirical performance of the APT with that of CAPM	2 factors for bond portfolios

Author (publication)	Statistical or theoretical methodology	the number of factors concluded or used
Chang & Lewellen (JFR Spring 85)	.zero beta portfolio .Applied to evaluate mutual fund performance	7 factors
DFGG (JBF 3/85a)	.MLFA .Common FA	$k = f(n, t)$ (+, +)
Raveh (JBF 6/85)	Literature Review on common factor analysis	(1) support the conclusion of DFGG (2) cast a doubt on the usefulness of conven- tional factor analysis for the APT testing
DFGG (JF 7/85b)	.MLFA .X ² -test	. $k_p = f(n, t)$.X ² (RP ₀) = f(n, t)
Trzcinka (JF 6/86)	.sampling theory of Eigenvalues .Eigenvalue Trace test	.1 dominant factor

Note: * denotes an unpublished dissertation

** represents a working paper

APPENDIX B
MEASUREMENT ERROR REVISITED

From equation (4-14)

$$R_2 = b_0 + b_1 R_1^* + b_2 R_b^* + e_2$$

$$\text{Let } x = [1, R_1^*, R_b^*]$$

$$B = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \quad \text{and} \quad y = [R_2]$$

then equation (4-14) can be written

$$y = \hat{x}B + \hat{e}$$

where the underlying true relationship is

$$y = xB + e \tag{B.1}$$

From (A.1),

$$\begin{aligned} e &= (x - \hat{x}) B + e \\ B &= (\hat{x}'\hat{x})^{-1} \hat{x}' y \\ &= (\hat{x}'\hat{x})^{-1} \hat{x}' (xB + e) \\ &= B + (\hat{x}'\hat{x})^{-1} \hat{x}' [(x - \hat{x}) B + e] \end{aligned} \tag{B.2}$$

$$\text{plim } \hat{B} = B \left[1 - \frac{\text{Var}(\hat{x} - x)}{\text{Var}(\hat{x})} \right] \tag{B.3}$$

Maximum likelihood factor analysis provides a complete maximum-likelihood solution for the total set of variables. Thus $\text{Var}(\hat{x} - x)$ in (B.3) can be ignored in large samples. Hence $\hat{B}$ is

asymptotically consistent. However, the problem lies in variance or standard error term form (B.2).

$$\begin{aligned}
 (\hat{B} - B)(\hat{B} - B) &= (\hat{x}'\hat{x})^{-1} \hat{x}'[(x - \hat{x})BB'(x - \hat{x}) + B'(x - \hat{x})'e \\
 &\quad + e'(x - \hat{x})B + ee']\hat{x}(\hat{x}'\hat{x})^{-1} \\
 &= (\hat{x}'\hat{x})^{-1} \hat{x}'[(x - \hat{x})BB'(x - \hat{x})' + \sigma_e^2] \hat{x}(\hat{x}'\hat{x})^{-1}
 \end{aligned}
 \tag{B.4}$$

Hence, $(\hat{x}'\hat{x})^{-1} \hat{x}'[(x - \hat{x})BB'(x - \hat{x})]\hat{x}(\hat{x}'\hat{x})^{-1}$ term remains as the inconsistent portion of the variance. Since factors are linear combinations of each economic variables (θ)

$$\begin{aligned}
 x &= z\theta \\
 \hat{x} &= z\hat{\theta}
 \end{aligned}
 \tag{B.5}$$

where z is an index for linear combination.

Therefore, $E[z(\theta - \hat{\theta})BB'(\theta - \hat{\theta})'z']$ term decides the magnitude of inconsistent portion of variance. However, $E[z(\theta - \hat{\theta})BB'(\theta - \hat{\theta})'z']$ term can be safely assumed to be ignorable because $(x - \hat{x})$ term is small, $(\hat{x}'\hat{x})^{-1}$ term is large, and thus whole term $(\hat{x}'\hat{x})^{-1} \hat{x}'[(x - \hat{x})BB'(x - \hat{x})]\hat{x}(\hat{x}'\hat{x})^{-1}$ in (B.4) is expected to be very small. More interested reader should consult Durbin (1954).

APPENDIX C
AN ALTERNATIVE DERIVATION OF FACTOR SCORE

This derivation is due to Berges (1982). Denote F_1 and F_2 to be the true factor scores being estimated in portfolio 1 and 2 associated with an orthogonal transformation as

$$F_1 = g_2 F_2 \quad (C.1)$$

F_1 and F_2 are unknown but estimated as

$$\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (C.2)$$

where it can be assumed with random grouping of securities

$$E(u_i' u_i) = V$$

$$E(u_i' u_j) = 0 \text{ for } i \neq j$$

Substitution of (C.2) into (C.1) leads to

$$\hat{F}_1 = g_2 (\hat{F}_2 - u_2) + u_1 \quad (C.3)$$

Estimating g_2 by OLS results in:

$$\hat{g}_2 = g_2 - 2(\hat{F}_2' \hat{F}_2 + u_2' u_2)^{-1} (u_2' u_2) f_2 \quad (C.4)$$

In order to estimate the factor score of portfolio 2 which is rotated to congruence with that of portfolio 1, denoted as

$F_1(2)$, substitute (C.4) in (C.3)

$$\begin{aligned}\hat{F}_1(2) &= \hat{g}_2' \hat{F}_2 \\ &= [I - 2(\hat{F}_2' \hat{F}_2 + u_2' u_2)^{-1} (u_2' u_2)] g_2' \hat{F}_2\end{aligned}\quad (C.5)$$

The pooled estimator of F_1 from the 2 portfolios can be expressed as

$$\begin{aligned}\hat{\bar{F}}_1 &= (\hat{F}_1 + \hat{F}_1(2))/2 \\ &= F_1 = (1/2)(U_1 + g_2' U_2 - C_2^{-1} U_2' g_2 U_2)\end{aligned}$$

$$\text{where } C_2 = 2(\hat{F}_2' \hat{F}_2 + u_2' u_2) \quad (C.6)$$

In general, the pooled estimator from N portfolios can be written

$$\begin{aligned}\hat{\bar{F}}_1 &= (1/N)(\hat{F}_1 + \hat{F}_1(2) + \dots + \hat{F}_1(N)) \\ &= F_1 + (1/N)[u_1 + \sum_j (g_j u_j - C_j^{-1} u_j' u_j (F_1 + g_j u_j))]\end{aligned}\quad (C.7)$$

Denoting the estimation error in $\hat{\bar{F}}_1$ as ER

$$ER = (1/N)[u_j + \sum_j (I - C_j^{-1} u_j' u_j) g_j u_j] - 1/N(\sum_j C_j^{-1} u_j' u_j F_j)$$

Thus, $\lim ER1$ can be written as

$$\begin{aligned}\lim ER1 &= \lim (1/N)[u_1 + \sum_j (I - C_j^{-1} u_j' u_j) g_s u_j] \\ &= 0 + (1/N) \sum_j (I - C_j^{-1} u_j' u_j) g_j \lim u_j = 0\end{aligned}$$

Hence

$$\begin{aligned}
 \lim F_1 &= F_1 - (1/N) \sum_j \lim C_j^{-1} u_j u_j' F_1 \\
 &= F_1 - (I + V)^{-1} V F_1 \\
 &= [I - (I + V)^{-1}] F_1 = h F_1
 \end{aligned} \tag{C.8}$$

To put (C.8) another way, as the number of portfolios becomes large, the pooled estimator F_1 asymptotically approaches a certain transformation of F_1 .

Thus F_1 can be retransformed to have an identity covariance matrix as F_1 with the certain h .

$$\text{Proof. } F_1 F_1' = h F_1 F_1' h' = h h' = m \tag{C.9}$$

Premultiplying F_1 by $m^{-1/2}$ and define

$$F_i^* = m^{-1/2} h F_1$$

such that

$$F_1^* F_1' = m^{-1/2} h F_1 F_1' h' m^{-1/2}$$

$$= m^{-1/2} m m^{-1/2} = I$$

Q.E.D.

REFERENCES

- Admati, A. F., and Pfleiderer, P., "Interpreting the Factor Risk Premia in the Arbitrage Pricing Theory." *Journal of Economic Theory* 35 (1985), 191-195.
- Anderson, T.W., "Asymptotic Theory for a Principal Component Analysis." *Annals of Mathematical Statistics* 34 (1963), 122-148.
- Banz, R., "The Relationship between Return and Market Value of Common Stocks." *Journal of Financial Economics* 9 (1983), 3-18.
- Bartlett, M.S., "Tests of Significance in Factor Analysis." *British Journal of Mathematical and Statistical Psychology* 3 (1950), 77-85.
- Berges, L.A., "The Arbitrage Pricing Theory: Estimation and Applications." Unpublished Ph.D. Dissertation, Purdue University, West Lafayette (1982).
- Black, F., "Capital Market Equilibrium with Restricted Borrowing." *Journal of Business* 45 (1972), 444-445.
- Bower, D.H., Bower, R.S., and Logue, D.E., "Arbitrage Pricing Theory and Utility Stock Returns." *Journal of Finance* 39 (1984), 1041-1054.
- Brennan, Michael J., "Discussion." *Journal of Finance* 36 (1981), 352-353.
- Brown, S. and Weinstein, M., "The Bilinear Asset Pricing Paradigm: A new Test of the Arbitrage Pricing Model." *Journal of Finance* 38 (1983), 711-743.
- Chamberlain, G., "Funds, Factors, and Diversification in Arbitrage Pricing Models." *Econometrica* 51 (1983), 1305-1323.
- Chamberlain, G., and Rothschild, M., "Arbitrage and Mean-Variance Analysis on Large Asset Markets." *Econometrica* 51 (1983), 1281-1304.
- Chan, K.C, Chen, N.F., and Hsieh, D.A., "An Exploratory Investigation of the Firm Size Effect." *Journal of Financial Economics* 14 (1985), 451-471.

- Chang, E.C., and Lewellen, W.G., "An Arbitrage Pricing Approach to Evaluating Mutual Fund Performance." *Journal of Financial Research* 8 (1985), 15-30.
- Chen, Nai-fu, "Some Empirical Tests of the Theory of Arbitrage Pricing." *Journal of Finance* 38 (1983), 1393-1414.
- Chen, N.F., and Ingersoll J.E., "Exact Pricing in Linear Factor Models with Finitely Many Assets: A Note." *Journal of Finance* 38 (1983), 985-988.
- Chen, N.F., Roll, R., and Ross, S., "Economic Forces and the Stock market: Testing the APT and Alternative Asset Pricing Theories." Working paper, University of Chicago (1983)
- Cho, D.C., "On Testing the Arbitrage Pricing Theory: Inter-battery Factor Analysis." *Journal of Finance* 39 (1984), 1485-1502.
- Cho, D.C., Elton, E.J., and Gruber, M.J., "On the Robustness of the Roll and Ross Arbitrage Pricing Theory." *Journal of Financial and Quantitative Analysis* 19 (1984), 1-10.
- Connor, G., "A Unified Beta Pricing Theory." *Journal of Economic Theory* 34 (1984), 13-31.
- Connor, G., and Korajczyk, R.A., "Performance Measurement with the Arbitrage Pricing Theory." *Journal of Financial Economics* 15 (1986), 373-394.
- Dhrymes, P.J., "The Empirical Relevance of Arbitrage Pricing Models." *Journal of Portfolio Management* 11 (1984), 35-44.
- Dhrymes, P.J., Friend, I., and Gultekin, N.B., "A Critical Reexamination of the Empirical Evidence on the Arbitrage Pricing Theory." *Journal of Finance* 39 (1984), 323-346
- Dhrymes, P.J., Friend, I., Gultekin, M.N., and Gultekin N.B., "An Empirical Examination of the Implication of Arbitrage Pricing Theory." *Journal of Banking and Finance* 9 (1985a), 73-99.
- Dhrymes, P.J., Friend, I., Gultekin, M.N., and Gultekin, N.B., "New Test of the APT and Their Implications." *Journal of Finance* 40 (1985b), 659-674.
- Durbin, J.M., "Errors in Variables." *Review of the International Statistical Institute* 22 (1954), 23-32.
- Dybvig, P.H., "An Explicit Bound on Individual Assets' Deviations from APT Pricing in a Finite Economy." *Journal of Financial Economics* 12 (1983), 483-496.
- Dybvig, P.H., and Ross, S.A., "Yes, the APT is Testable." *Journal of Finance* 40 (1985), 1173-1188.

- Huang, R.D., and Jo, H., "Tests of Market Models: Heteroskedasticity or Misspecification?" Working Paper, University of Florida, Gainesville (February 1986a).
- Huang, R.D., and Jo, H., "A Factor Analytic Approach to Foreign Exchange Speculation." forthcoming at the Proceedings of the American Statistical Association, Business and Economics Statistics Section. Washington: American Statistical Association (1986b).
- Huberman, Gur., "A Simple Approach to Arbitrage Pricing Theory." *Journal of Economic Theory* 28 (1982), 183-191.
- Ingersoll, J.E., "Some Results in the Theory of Arbitrage Pricing." *Journal of Finance* 39 (1984), 1021-1039.
- Jo, H., "The Factors in the Arbitrage Pricing Theory: An Empirical Examination." Working Paper, University of Florida, Gainesville (1984).
- Jobson, J.D., "A Multivariate Linear Regression Test for the Arbitrage Pricing Theory." *Journal of Finance* 37 (1982), 1037-1042.
- Johnson, R.M., "On a Theorem Stated by Eckart and Young." *Psychometrika* 28 (1963), 259-263.
- Jordan, B., "The Arbitrage Model of Security Returns: An Empirical Evaluation." Unpublished Ph.D. Dissertation, University of Florida, Gainesville (1984).
- Joreskog, K.G., "Some Contributions to Maximum Likelihood Factor Analysis." *Psychometrika* 32 (1967), 443-482.
- Kaiser, H.F., "The Varimax Criterion for Analytic Rotation in Factor Analysis." *Psychometrika* 23 (1958), 187-200.
- Keim, D.B., "Size-related Anomalies and Stock Market Seasonality: Further Empirical Evidence." *Journal of Financial Economics* 12 (1983), 13-32.
- Kraus, A., "Discussion." *Journal of Finance* 40, (1985), 674-675.
- Kruskal, J.B., "Factor Analysis and Principal Components." In W.H. Kruskal and J.M. Tanur, *International Encyclopedia of Statistics*. The Free Press (1978), 307-337.
- Kryzanowski, L. and To, M.C., "General Factor Models and the Structure of Security Returns." *Journal of Financial and Quantitative Analysis* 18 (1983), 31-52.
- Lawley, C.N., and Maxwell, A.D., *Factor Analysis as a Statistical Method*, New York: American Elsevier Publishing Company, Inc. (1971).

- Eckart, C., and Young, G., "The Approximation of One Matrix by Another of Lower Rank." *Psychometrika* 1 (1936), 211-218.
- Elton, E., Gruber, M., and Rentzler, J., "The Arbitrage Pricing Model and Returns on Assets Under Uncertain Inflation." *Journal of Finance* 38 (1983), 525-537.
- Fama, Eugene, *Foundations of Finance*, New York: Basic Books, Inc., (1976).
- Fama, E.F., and MacBeth, J.D., "Risk, Return, and Equilibrium: Empirical Tests." *Journal of Political Economy* 81 (1973), 607-636.
- Fogler, H.R., "Common Sense on CAPM, APT, and Correlated Residuals." *Journal of Portfolio Management* 9 (1982), 20-28.
- Fogler, H.R., Kose, J., and Tipton, J., "Three Factors, Interest Rate Differentials and Stock Groups." *Journal of Finance* 36 (1981), 323-335.
- Friend, I., "Discussion." *Journal of Finance* 36 (1981), 350-352.
- Gehr, Adam, Jr., "Some Tests of the Arbitrage Pricing Theory." *Journal of the Midwest Finance Association* (1978), 91-105.
- Gibbons, M.R., "Empirical Examination of the Return generating Process of the Arbitrage Pricing Theory." Working Paper, Stanford University, Stanford, California (1983).
- Green, B.F., "The Orthogonal Approximation of an oblique structure in factor analysis." *Psychometrika* 17 (1952), 429-440.
- Grinblatt, M., and Titman, S., "Factor Pricing in a Finite Economy." *Journal of Financial Economics* 12 (1983), 497-507.
- Grinblatt, M., and Titman, S., "The Relation Between Mean Variance Efficiency and Arbitrage Pricing." Working Paper, University of California, Los Angeles (1985a).
- Grinblatt, M., and Titman, S., "Approximate Factor Structures: Interpretations and Implications for Empirical Tests." *Journal of Finance* 40 (1985b), 1367-1373.
- Gultekin, N.B., and Rogalski, R.J., "Government Bond Returns, Measurement of Interest Rate Risk, and the Arbitrage Pricing Theory." *Journal of Finance* 40 (1985), 43-61.
- Harmon, H., *Modern factor Analysis*, Chicago: University of Chicago Press, (1976).
- Hausman, J.A., "Specification Tests in Econometrics." *Econometrica* 46 (1978), 1251-1271.

- Lehmann, B. and Modest, D., "Mutual Fund Performance Evaluation: A comparison of Benchmarks and Benchmark Comparisons." Working paper, Columbia University, New York (1985).
- Levy, H., "Equilibrium in an Imperfect Market: A Constraint on the number of Securities in the Portfolio." *American Economic Review* 68 (1978), 643-658.
- Levy, H, and Sarnat, M., *Portfolio and Investment Selection: Theory and Practice*. Englewood Cliffs, New Jersey: Prentice-Hall International (1984).
- Lintner, J. "Security Prices, Risk, and the Maximal Gains from Diversification." *Journal of Finance* 20 (1965), 587-615.
- Livingston, Miles, "Industry Movements of Common Stocks." *Journal of Finance* 32 (1977), 861-874.
- Markowitz, H.M., "Portfolio Selection." *Journal of Finance* 7 (1952), 77-91.
- Markowitz, H.M., *Portfolio Selection: Efficient Diversification of Investments*. New York: John Wiley & Sons Inc., (1959).
- Merton, R.C., "An Analytic Derivation of the Efficient Portfolio Frontier." *Journal of Financial and Quantitative Analysis* 7 (1972), 1852-1857.
- Merton, R.C., "An Intertemporal Capital Asset Pricing Model." *Econometrica* 41 (1973), 867-887.
- Morrison, D.F., *Multivariate Statistical Methods*, New York: McGraw-Hill Book Company (1976).
- Mossin, J., "Equilibrium in a Capital Asset Market." *Econometrica* 34 (1966), 768-783.
- Oldfield, G., and Rogalski, R.J., "Treasury Bill Factors and Common Stock Returns." *Journal of Finance* 36 (1981), 337-350.
- Pari, Robert A. "An Empirical Test of Ross's Arbitrage Pricing Theory." Unpublished Ph.D. Dissertation, Virginia Polytechnic Institute and State University, Blacksburg (1980).
- Pari, R.A., and Chen, S.N., "An Empirical Test of the Arbitrage Pricing Theory." *Journal of Financial Research* 7 (1984), 121-130.
- Pettway, R.H., "On the Use of B in Regulatory Proceedings: An Empirical Examination." *Bell Journal of Economics* 8 (1978), 239-248.

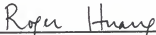
- Pfleiderer, P., "A Short Note on the Similarities and the Differences Between the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT)." Working Paper, Stanford University, Stanford, California (1983).
- Ravesh, A., "A Note on Factor Analysis and Arbitrage Pricing Theory." *Journal of Banking and Finance* 9 (1985), 317-321.
- Reinganum, Marc R., "The Arbitrage Pricing Theory: Some Empirical Results." *Journal of Finance* 36 (1981), 313-321.
- Roll, Richard, "A Critique of Asset Pricing Theory's Tests: Part I: On Past and Potential Testability of the Theory." *Journal of Financial Economics* 4 (1977), 129-176.
- Roll, Richard, and Ross, Stephen A., "An Empirical Investigation of the Arbitrage pricing Theory." *Journal of Finance* 35 (1980), 1073-1103.
- Roll, R., and Ross, Stephen A., "A Critical Reexamination of the Empirical Evidence on the Arbitrage Pricing Theory: A Reply." *Journal of Finance* 39 (1984), 347-350.
- Ross, Stephen A., "The Arbitrage Theory of Capital Asset Pricing." *Journal of Economic Theory* 13 (1976), 341-360.
- Ross, Stephen A., "Return, Risk, and Arbitrage." In *Risk and Return in Finance*, I. Friend and J. Bicksler (eds.), Vol 1, Cambridge, MA: Ballinger (1977), 189-218.
- Ross, Stephen A., "The Current Status of the Capital Asset Pricing Model." *Journal of Finance* 33 (1978), 885-901.
- Schonemann, P.H., "A Generalized Solution of the Orthogonal Procrustes Problem." *Psychometrika* 31 (1966), 1-10.
- Schwert, G.W., "Size and Stock Returns, and Other Empirical Regularities." *Journal of Financial Economics* 12 (1983), 3-12.
- Shanken, J., "The Arbitrage Pricing Theory: Is it Testable?" *Journal of Finance* 37 (1982), 1129-1140.
- Shanken, J., "Multi-Beta CAPM or Equilibrium-APT?: A Reply." *Journal of Finance* 40 (1985), 1189-1196.
- Sharpe, W.F., "Capital Asset Prices: A Theory of Equilibrium Under Conditions of Risk." *Journal of Finance* 19 (1964), 425-442.
- Stambaugh, Robert F., "Missing Assets, Measuring the Market and Testing the Capital Asset Pricing Model." Unpublished Ph.D. Dissertation, University of Chicago, Chicago (1981).

- Ten Berge, J.M.F., "A Joint Treatment of Varimax Rotation, and the Problem of Diagonalizing symmetric Matrices Simultaneously in the least-Squares Sense." *Psychometrika* 49 (1984), 347-358.
- Tinic, S.M., and West, R.R., "Risk and Return: January vs. the Rest of the Year." *Journal of Financial Economics* 13 (1984), 561-574.
- Tinic, S.M., and West, R.R., "Risk. Return, and Equilibrium: A Revisit." *Journal of Political Economy* 91 (1986), 126-147.
- Tobin, J. "Liquidity Preference as Behavior Towards Risk." *Review of Economic Studies* 25 (1958), 65-86.
- Trzcinka, D., "On the Number of Factors in the Arbitrage Pricing Model." *Journal of Finance* 41 (1986), 347-368.
- Velicer, W.F., Peacock, A.C., and Jackson, D.N. "A Comparison of Component and Factor Patterns: A Monte Carlo Approach." *Multivariate Behavioral Research* 17 (1982), 371-388.
- White, H., "A Heteroskedasticity-Consistent Covariance Matrix Estimator and A Direct Test for Heteroskedasticity." *Econometrica* (1980), 817-838.
- Wilcoxon, F., "Individual Comparisons by Ranking Methods." *Biometric Bulletin* 1 (1945), 80-83.

BIOGRAPHICAL SKETCH

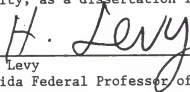
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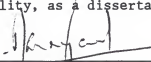
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